HIGHER MATHS

Differentiation

Notes with Examples

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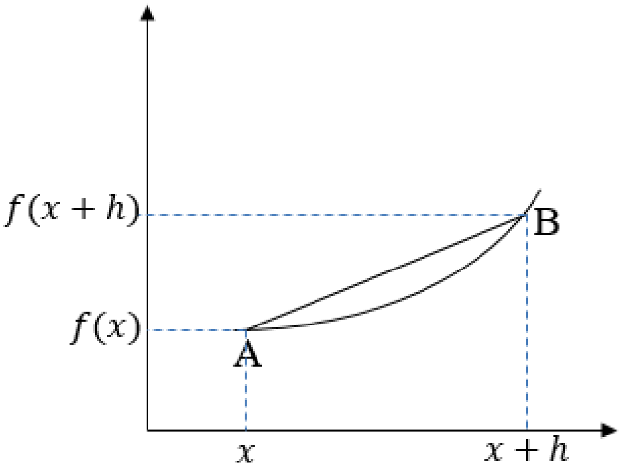
What is Differentiation?

You can find the gradient of a line joining two points on a curve using the gradient formula.



$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(x+h) - f(x)}{(x+h) - x}$$
$$= \frac{f(x+h) - f(x)}{h}$$

If B is then moved closer to A (making h smaller and smaller), the line AB will become nearer to the tangent at A. As h tends to 0, the gradient of AB tends to a limit. This limit is the gradient of the tangent at A.

This can be written as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is the formal definition of $f'(x)$ the derivative of $f(x)$. The derived function represents the rate of change of the function and can be used to find the instantaneous gradient at any point on the curve.

The process of deriving $f'(x)$ from $f(x)$ is called differentiation. The method above is called differentiating from first principles.

Notation

There are three types of notation for differentiation.

We can use function notation. $f(x)$ for our original function and $f'(x)$ for the derivative.

In Leibniz notation given $y = \cdots$, the derivative of y with respect to x is written $\frac{dy}{dx}$

When starting with an expression we can also use $\frac{d}{dx}$ to indicate the derivative.

Rules for Differentiating

Basic Differentiation

The basic rule allows you to differentiate powers of x.

$$\text{If } f(x) = x^n \quad \text{then} \quad f'(x) = nx^{n-1}$$

Examples

D-01 For each of the following, find the derivative

1. $f(x) = x^5$ 2. $y = x^{-3}$ 3. x^{10}

4. $y = x^{3/2}$ 5. x 6. $f(x) = 5$

1. $f(x) = x^5$
$f'(x) = 5x^4$

2. $y = x^3$
$\frac{dy}{dx} = 3x^2$

3. $\frac{d}{dx}(x^{10})$
$= 10x^9$

4. $y = x^{3/2}$
$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$

5. $\frac{d}{dx}(x)$
$= 1$

6. $f(x) = 5$
$f'(x) = 0$

Notice that the derivative of a constant term will **always be zero**.

Differentiating Terms with Coefficients

This rule can be extended to multiples of powers of x.

$$\text{If } f(x) = ax^n \quad \text{then} \quad f'(x) = anx^{n-1}$$

Examples

D-02 Find the derivative of each of the following:

1. $f(x) = 3x^2$ 2. $f(x) = 7x^5$

1. $f(x) = 3x^2$
$f'(x) = 2 \times 3x$
$= 6x$

2. $f(x) = 7x^5$
$f'(x) = 5 \times 7x^4$
$= 35x^4$

Differentiating Multiple Terms

This rule can be extended to multiples and sums or differences of powers of x.

> If $f(x) = ax^n + bx^m$ then $f'(x) = anx^{n-1} + bmx^{m-1}$

Examples

D-03 Find the derivative of each of the following:

1. $f(x) = 2x^2 - 3x + 4$
2. $f(x) = x^4 + 3x^3 - 5x$

$f(x) = 2x^2 - 3x + 4$
$= 4x - 3$

2. $f(x) = x^4 + 3x^3 - 5x$
$f'(x) = 4x^3 + 9x^2 - 5$

Preparing to Differentiate

It is often necessary to use the index rules to prepare a function for differentiation. The differentiation rules will only work with powers of x. In order to differentiate, you must be able to convert a function involving roots, fractions, brackets etc. into a sum or difference of powers of x.

If necessary, use the index rules to give the derivative using positive indices. There are normally no marks awarded for changing the form of a derivative once you have obtained it. However, you often have to go further and evaluate the derivative at a point and will find this easier using positive indices.

Examples

D-04 Differentiate:

1. $f(x) = \frac{1}{x^5}$
2. $f(x) = 2\sqrt{x}$
3. $f(x) = \frac{3}{4x^{-3}}$
4. $f(x) = \frac{x}{4} + \frac{6}{\sqrt[3]{x}}$

1. $f(x) = \frac{1}{x^5}$
$= x^{-5}$
$f'(x) = -5x^{-4}$
$= \frac{-5}{x^4}$

2. $f(x) = 2\sqrt{x}$
$= 2x^{1/2}$
$f'(x) = x^{-1/2}$
$= \frac{1}{\sqrt{x}}$

3. $f(x) = \frac{3}{4x^{-3}}$
$= \frac{3x^3}{4}$
$f'(x) = \frac{9x^2}{4}$

4. $f(x) = \frac{1}{x} + \frac{6}{\sqrt[3]{x}}$
$= x^{-1} + 6x^{-1/3}$
$f'(x) = -x^{-2} - 2x^{-4/3}$
$= -\frac{1}{x^2} - \frac{2}{\sqrt[3]{x^4}}$

Derivatives of products and quotients

We can also find the derivatives of more complicated expressions. Products can be prepared for differentiation by removing brackets. Quotients must be split into a sum or difference of powers of x before differentiating.

Examples

D-05 Differentiate:

1. $f(x) = (x-3)(x+5)$

2. $f(x) = \sqrt{x}(x^3 + \sqrt{x})$

3. $f(x) = \frac{x^2+2x+1}{x}$

4. $f(x) = \frac{x+1}{\sqrt{x}}$

1. $f(x) = x^2 + 5x - 3x - 15$
$= x^2 + 2x - 15$
$f'(x) = 2x + 2$

2. $f(x) = x^{1/2}(x^3 + x^{1/2})$
$= x^{7/2} + x$
$f'(x) = \frac{7}{2}x^{5/2} + 1$
$= \frac{7}{2}\sqrt{x^5} + 1$

3. $f(x) = \frac{x^2}{x} + \frac{2x}{x} + \frac{1}{x}$
$= x + 2 + x^{-1}$
$f'(x) = 1 - x^{-2}$
$= 1 - \frac{1}{x^2}$

4. $f(x) = \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}$
$= \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}}$
$= x^{1/2} + x^{-1/2}$
$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$
$= \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

Differentiating with Respect to Other Variables

All the examples so far have asked us to differentiate with respect to x, but the rules still apply when differentiating with respect to another variable. This is usually when modelling real-life problems.

Examples

D-06 Differentiate $3u^4 - 5u$ with respect to u.

$$\frac{d}{du}\left(3u^4 - 5u\right)$$
$$= 12u^3 - 5$$

D-07 Differentiate qx^4 with respect to q.

$$\frac{d}{dq}\left(qx^4\right)$$
$$= \frac{d}{dq}\left(x^4 q\right)$$
$$= x^4$$

D-08 Given $A(r) = \pi r^2$, find $A'(r)$.

$$A(r) = \pi r^2$$
$$A'(r) = 2\pi r$$

Rate of Change

Evaluating the derived function for a given value of x gives the **rate of change** of the function at that point.

Key point to note is that velocity is the rate of change of displacement and acceleration is the rate of change of velocity.

Examples

D-09 The value of an investment is calculated using $V(t) = \sqrt{t^3}$, where V is the value and t is the time in years. Calculate the growth rate (rate of change) of the investment after 9 years.

$$V(t) = \sqrt{t^3}$$
$$= t^{3/2}$$
$$V'(t) = \frac{3}{2}t^{1/2}$$
$$= \frac{3}{2}\sqrt{t}$$
$$V'(9) = \frac{3}{2}\sqrt{9}$$
$$= \frac{9}{2}$$

D-10 A ball is thrown so that its displacement s after t seconds is given by $s(t) = 12t - 5t^2$. Find its velocity after 1 second.

$$s(t) = 12t - 5t^2$$
$$s'(t) = 12 - 10t^2$$
$$s'(t) = v(t)$$
$$v(1) = 12 - 10(1)$$
$$= 2 \text{ m/s}$$

Equation of Tangents

We can find the gradient of a curve, which is also the gradient of the tangent to the curve at a given point by evaluating the derivative.

For $y = \ldots$ the value of $\frac{dy}{dx}$ when $x = a$ is the gradient of the tangent at a

For $f(x) = \ldots$ the gradient of the tangent at $x = a$ is given by $f'(a)$

To find the equation of a tangent to the curve at the point $x = a$:

- Use the derivative to find the gradient
- Substitute $x = a$ into $y = f(x)$ to find the coordinates of the point on the curve.
- Substitute these values into the straight line equation $y - b = m(x - a)$

Examples

D-11 Find the equation of the tangent to the curve $y = x^2 - 5x + 3$ when $x = 2$.

$y = x^2 - 5x + 3$

$\frac{dy}{dx} = 2x - 5$

when $x = 2$

$y = (2)^2 - 5(2) + 3$
$= 4 - 10 + 3$
$= -3$

$\boxed{(2, -3)}$

$\frac{dy}{dx} = 2(2) - 5$
$= 4 - 5$
$= -1$

$\boxed{m = -1}$

tangent

$y - b = m(x - a)$
$y + 3 = -1(x - 2)$
$y + 3 = -x + 2$
$y = -x - 1$

D-12 Find the equation of the tangent to y = $\sqrt{x^3}$ at $x = 9$.

$y = \sqrt{x^3}$
$= x^{3/2}$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$
$= \frac{3}{2}\sqrt{x}$

when $x = 9$

$y = \sqrt{9^3}$
$= 27$

$\boxed{9, 27}$

$\frac{dy}{dx} = \frac{3}{2}\sqrt{9}$
$= \frac{9}{2}$

$\boxed{m = \frac{9}{2}}$

tangent

$y - b = m(x - a)$
$y - 27 = \frac{9}{2}(x - 9)$

$2y - 54 = 9x - 81$

$2y - 9x + 27 = 0$

D-13 Find the points of contact on the curve $y = x^3 - 4x$ for tangents with gradient 8.

$y = x^3 - 4x$

$\frac{dy}{dx} = 3x^2 - 4$

$\frac{dy}{dx} = m$, $m = 8 \Rightarrow 3x^2 - 4 = 8$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

when $x = 2$ $\quad y = (2)^3 - 4(2)$

$= 0$ $\quad \boxed{\text{PoC } (2,0)}$

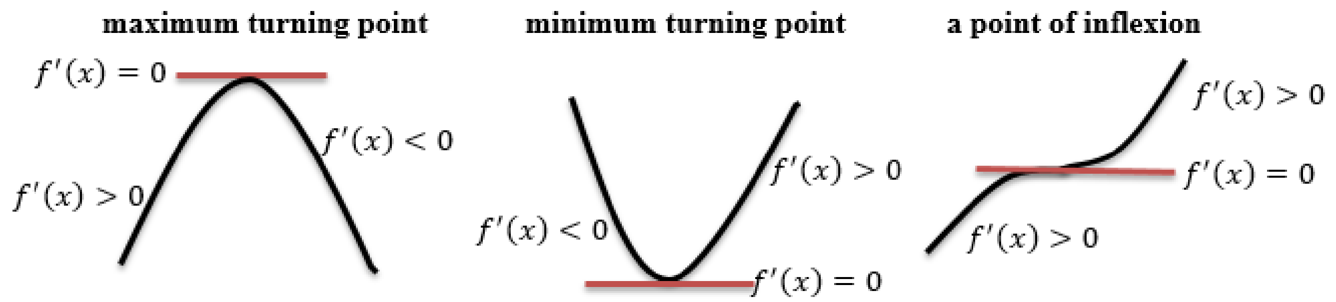
when $x = -2$ $\quad y = (-2)^3 - 4(-2)$

$= 0$ $\quad \boxed{\text{PoC } (-2,0)}$

Stationary Points

On the graph of $y = f(x)$, if $f'(a) = 0$ the curve has gradient 0 at $x = a$ and so is horizontal at that point. You say the graph has **a stationary point** at $(a, f(a))$.

There are different kinds of stationary point. The nature of a stationary point depends on the gradient on either side of it. There are **three** kinds of stationary points.



A point of inflexion can be either rising or falling. A falling point of inflexion has a negative gradient on either side of the stationary point, a rising point of inflexion has a positive gradient on either side of the stationary point.

You can determine the **nature** of the stationary point by considering the value of $f'(x)$ on either side of it. This is normally done using a **nature table**. The derivative is found for a value of x on either side of each solution to $f'(x) = 0$. The table records the sign (positive or negative) of these derivatives not the actual value.

Examples

D-14 Find the stationary points on the curve $y = 2x^3 - 9x^2 + 12x$ and determine their nature.

$y = 2x^3 - 9x^2 + 12x$

$\frac{dy}{dx} = 6x^2 - 18x + 12$

For SP $\frac{dy}{dx} = 0$

$6x^2 - 18x + 12 = 0$

$6(x^2 - 3x + 2) = 0$

$6(x-2)(x-1) = 0$

$x = 2 \quad x = 1$

NT

x	$\xrightarrow{0}$	1	$\xrightarrow{1.5}$	2	$\xrightarrow{3}$
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
Shape	/	—	\	—	/

When $x = 1$ $\quad y = 2(1)^3 - 9(1)^2 + 12(1)$

$= 5$ $\quad \boxed{(1,5) \text{ MAX TP}}$

When $x = 2$ $\quad y = 2(2)^3 - 9(2)^2 + 12(2)$

$= 4$ $\quad \boxed{(2,4) \text{ MIN TP}}$

D-15 Find the stationary values of the function $f(x) = 4x^3 - x^4$ and determine their nature.

$f(x) = 4x^3 - x^4$

$f'(x) = 12x^2 - 4x^3$

For SP $f'(x) = 0$

$12x^2 - 4x^3 = 0$

$4x^2(3 - x) = 0$

$x = 0 \quad x = 3$

NT

x	$\xrightarrow{-1}$	0	$\xrightarrow{1}$	3	$\xrightarrow{4}$
$f'(x)$	$+$	0	$+$	0	$-$
Shape	/	—	/	—	\

When $x = 0$ $\quad y = 4(0)^3 - (0)^4$

$= 0$ $\quad \boxed{(0,0) \text{ RISING POINT OF INFLEXION}}$

When $x = 3$ $\quad y = 4(3)^3 - (3)^4$

$= 27$ $\quad \boxed{(3,27) \text{ MAX TP}}$

Increasing and Decreasing Functions

A function $f(x)$ is increasing at a point $x = a$ if $f'(a) > 0$.

A function $f(x)$ is decreasing at a point $x = a$ if $f'(a) < 0$.

To check where a function is increasing and decreasing we differentiate then evaluate the derivative using a **nature table**.

Examples

D-16 A function is given by $f(x) = x^3 - 3x^2$

On what interval is $f(x)$ (a) increasing (b) decreasing?

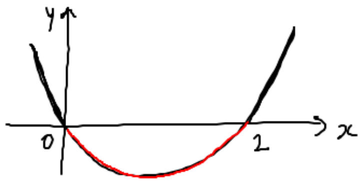
$f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x$

For SP $f'(x) = 0$ $\quad 3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0 \quad x = 2$



$f'(x) > 0 \qquad x < 0,\ x > 2$

INCREASING

$f'(x) < 0 \qquad 0 < x < 2$

DECREASING

NT

x	$\xrightarrow{-1}$	0	$\xrightarrow{1}$	2	$\xrightarrow{3}$
$f'(x)$	+	0	−	0	+
shape	/	—	\	—	/
	INC		DEC		INC

EITHER NATURE TABLE OR QUADRATIC INEQUALITY

D-17 Show that the function $f(x) = \frac{1}{3}x^3 + 2x^2 + 5x - 12$ is always increasing.

$f(x) = \frac{1}{3}x^3 + 2x^2 + 5x - 12$

$f'(x) = x^2 + 4x + 5$

$= (x+2)^2 + 5 - 4$

$= (x+2)^2 + 1$

MIN VALUE OF $f'(x) = 1$

SO $f(x)$ IS ALWAYS INCREASING

"always increasing" means we are looking at completing the square

Curve Sketching

To sketch a curve you must find all the significant points. That is where it cuts the axes and any stationary points. It can be helpful to consider what the graph does as x becomes large $x \to \infty$ and small $x \to -\infty$ to confirm your findings.

To sketch a curve:

- Find the intersection with the x and y axes.
- Find the stationary points and determine their nature as above.
- Check the behaviour of the graph for large and small values of x.
- Annotate the graph clearly with all significant points.

Examples

D-18 Sketch the curve $y = x^3 - x^2$

$y = x^3 - x^2$

$\frac{dy}{dx} = 3x^2 - 2x$

For SP $\frac{dy}{dx} = 0$ $\quad 3x^2 - 2x = 0$

$x(3x - 2) = 0$

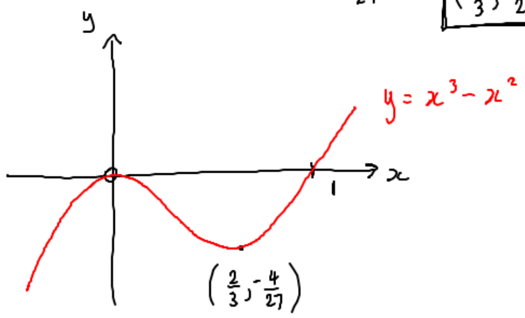
$x = 0 \quad x = \frac{2}{3}$

NT

x	$\xrightarrow{-1}$	0	$\xrightarrow{1/2}$	$\frac{2}{3}$	$\xrightarrow{1}$
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
Shape	/	—	\	—	/

when $x = 0$ $\quad y = (0)^3 - (0)^2$
$= 0$
$\boxed{(0, 0) \text{ MAX TP}}$

when $x = \frac{2}{3}$ $\quad y = \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2$
$= -\frac{4}{27}$
$\boxed{\left(\frac{2}{3}, -\frac{4}{27}\right) \text{ MIN TP}}$



CUTS X AXIS WHEN Y=0

$x^3 - x^2 = 0$

$x^2(x - 1) = 0$

$x = 0 \quad x = 1$

CUTS Y AXIS WHEN X=0

$y = (0)^3 - (0)^2$
$= 0$

Closed Intervals

In a closed interval the maximum and minimum values of a function are either at a stationary point or at an end point of an interval.

To find the maximum or minimum value evaluate the function at each end point and at each stationary value within the interval.

Examples

D-19 Find the maximum and minimum values of $f(x) = x(x-3)^2$ on the interval $-2 \leq x \leq 5$.

$$f(x) = x(x-3)^2$$
$$= x(x^2 - 6x + 9)$$
$$= x^3 - 6x^2 + 9x$$
$$f'(x) = 3x^2 - 12x + 9$$

For SP $f'(x) = 0$

$$3x^2 - 12x + 9 = 0$$
$$3(x^2 - 4x + 3) = 0$$
$$3(x-3)(x-1) = 0$$
$$x = 3 \quad x = 1$$

NT

x	$\xrightarrow{0}$	1	$\xrightarrow{2}$	3	$\xrightarrow{4}$
$f'(x)$	$+$	0	$-$	0	$+$
Shape	/	—	\	—	/

when $x = -2$ $\quad y = (-2)(-2-3)^2$
$= -50$

when $x = 1$ $\quad y = (1)(1-3)^2$
$= 4$

when $x = 3$ $\quad y = (3)(3-3)^2$
$= 0$

when $x = 5$ $\quad y = (5)(5-3)^2$
$= 20$

$$\boxed{\begin{array}{l} \text{MAX} = 20 \ @\ x = 5 \\ \text{MIN} = -50 \ @\ x = -2 \end{array}}$$

Graphs of Derived Functions

The derived function is sometimes called the gradient function as it gives the gradient value at any point on the curve $y = f(x)$.

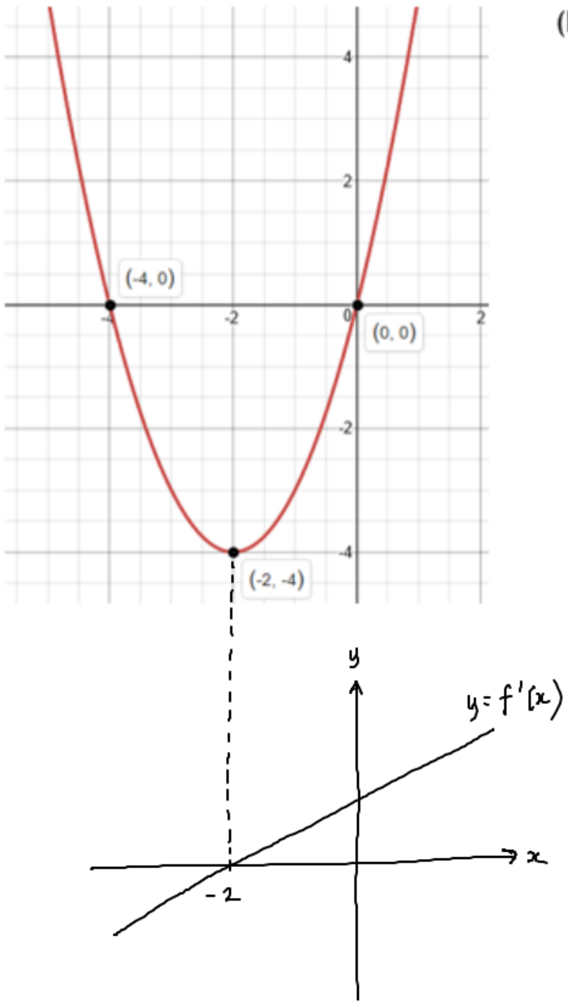
To sketch the derived function:

- Identify any stationary points on $y = f(x)$. They will give the zeros of the gradient function.
- Check where the function $y = f(x)$ is increasing (decreasing). The derivative is positive (negative) on those intervals, and so the gradient function will be above (below) the x-axis on those intervals.

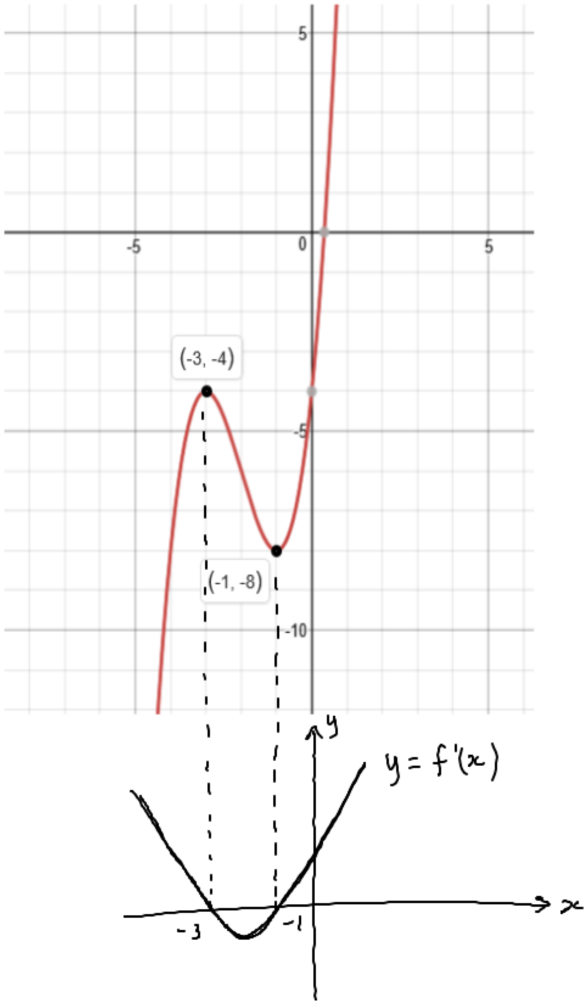
Examples

D-20 Sketch the graphs of $y = f'(x)$ for each of the following.

(a)



(b)



Optimisation

We can use differentiation to easily identify the maximum and minimum values of a function. When a real life problem is modelled with a function it is important to pay attention to the context. The process of funding the 'best' value for a particular situation is called **optimisation**.

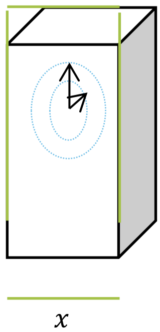
These problems often involve functions with two variables. You cannot differentiate a function with more than one variable. There will be an additional piece of information in the question which allows you to give the function in terms of one variable.

- Identify the function to be maximised or minimised. Look for clues in the wording.
- Differentiate this function and find where the derivative is zero.
- Check the nature of any stationary value.
- Check on what interval the function is defined and whether any stationary value(s) are inside that interval.
- Answer the question in **context**.

Examples

D-21 The glass front of a carriage clock has breadth x cm.

The jeweller has used 16cm of gold leaf to edge the perimeter of the glass.

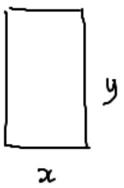


(a) Show that the area of glass on the front of the clock is given by $A(x) = 8x - x^2$

(b) Find the dimensions of the rectangle that give the maximum area.

(c) Calculate the maximum area.

(a)



PERIMETER $= 2x + 2y$

$2x + 2y = 16$

$2y = 16 - 2x$

$\boxed{y = 8 - x}$

AREA $= xy$

$= x(8 - x)$

$= 8x - x^2$

(b) $A(x) = 8x - x^2$

$A'(x) = 8 - 2x$

FOR SP $A'(x) = 0$ $\quad 8 - 2x = 0$

$x = 4$

NT

x	$\xrightarrow{1}$	4	$\xrightarrow{5}$
$A'(x)$	$+$	0	$-$
Shape	/	—	\

$x = 4$ MAXIMISES THE AREA

(c) $y = 8 - x$

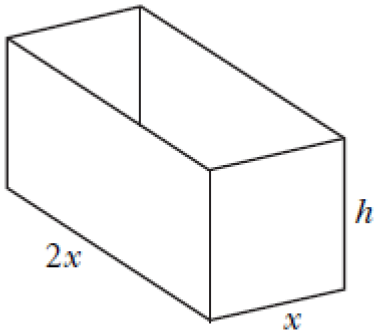
$y = 8 - 4$

$= 4$

MAX AREA $= 4 \times 4$

$= 16 \text{ cm}^2$

D-22 An open cuboid measures internally x units by $2x$ units by h units and has an inner surface area of 12 units^2.



(a) Show that the volume V units^3 of the cuboid is given by $V = \frac{2}{3}x(6 - x^2)$

(b) Find the exact value of x for which the volume is a maximum.

(a) $V = lbh$

$= 2x \times x \times h$

$= 2x^2 h$

$= 2x^2 \times \frac{1}{3}\left(\frac{6-x^2}{x}\right)$

$= \frac{2}{3}x(6-x^2)$

FRONT × 2 SIDE × 2 BASE

SURFACE AREA $= 2xh + 2(2xh) + 2x \times x$

$= 2xh + 4xh + 2x^2$

$= 6xh + 2x^2$

$6xh + 2x^2 = 12$

$6xh = 12 - 2x^2$

$h = \frac{12 - 2x^2}{6x}$

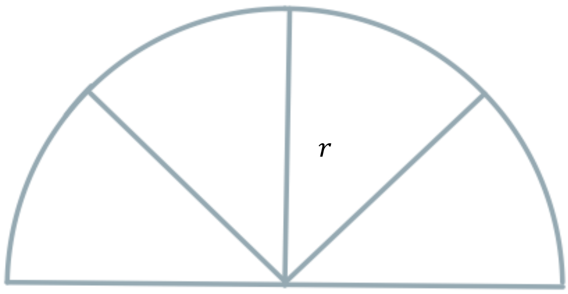
$= \frac{2}{6}\left(\frac{6 - x^2}{x}\right)$

$= \boxed{\frac{1}{3}\left(\frac{6-x^2}{x}\right)}$

D-23 A semi-circular window of radius r cm is divided into sectors, each of area 125 cm^2.

(a) Show that the perimeter of each sector is $P = 2(r + \frac{125}{r})$.

(b) Find the minimum value of P.



(a) AREA: $\frac{\pi r^2}{8} = 125$ $\quad \frac{\pi r}{8} = \frac{125}{r}$

ARC: $\frac{\pi d}{8} = \frac{2\pi r}{8} = \frac{\pi r}{4}$

PERIMETER: $2r + \frac{\pi r}{4}$

$= 2\left(r + \frac{\pi r}{8}\right)$

$= 2\left(r + \frac{125}{r}\right)$

Summary

RULES:

$f(x) = ax^n$

$f'(x) = nax^{n-1}$

"multiply by the power then decrease the power"

REMEMBER: Numbers differentiate to 0.

eg $y = 3x^4 + 5$

$\frac{dy}{dx} = 12x^3 + 0$

$= 12x^3$

$y \to \frac{dy}{dx}$

$f(x) \to f'(x)$

INCREASING / DECREASING FUNCTIONS:

$f'(x) > 0$ increasing

$f'(x) = 0$ stationary

$f'(x) < 0$ decreasing

STATIONARY POINTS:

Follow these steps:

1. find $f'(x)$
2. Set $f'(x) = 0$ as SP $f'(x) = 0$
3. Solve $f'(x) = 0$ usually by factorising
4. Sub x values from 3 into y= for coordinates
5. Nature table
6. State points and nature

PREPARING TO DIFFERENTIATE:

We can't differentiate roots, brackets or x as a denominator

$\sqrt[3]{x^2} \Rightarrow x^{2/3}$

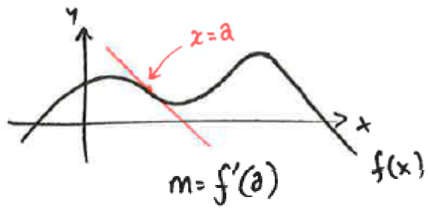
$x(x+2) \Rightarrow x^2 + 2x$

$\frac{3}{x^2} \Rightarrow 3x^{-2}$

THIS MUST BE DONE BEFORE DIFFERENTIATI

GRADIENT OF TANGENT TO A CURVE:

$f'(x) = m_{TAN}$ at x

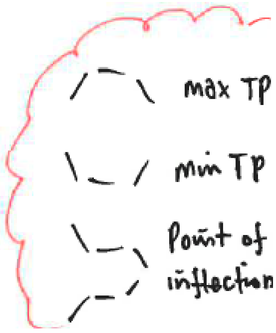


EQUATION OF TANGENT TO A CURVE:

Follow these steps:

1. Sub x value into f(x) to get coordinates of point of contact
2. Find $f'(x)$
3. Sub x value into $f'(x)$ to find m_{TAN}
4. Sub in PoC & m_{TAN} into $y - b = m(x - a)$

x	→	a	→
$f'(x)$	"+ or −"	0	"+ or −"
Shape	/ (+)	—	\ (−)



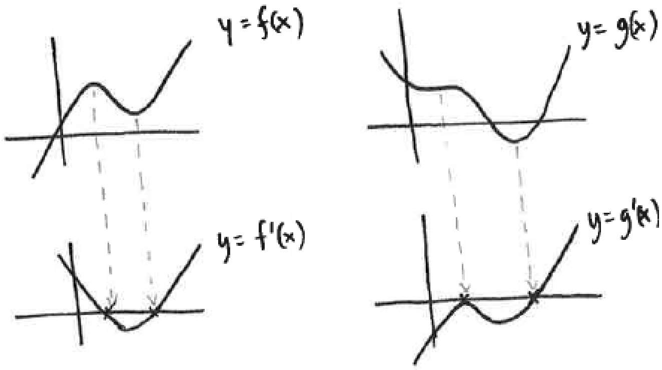
GRAPH OF DERIVED FUNCTION :

Sketch f(x) then f'(x) on a seperate graph below

f(x) SP → f'(x) cuts x axis

f(x) inc → f'(x) above x-axis

f(x) dec → f'(x) below x-axis



OPTIMISATION :

Finding a max or min value. First part of question is usually A/B level.
Second part is just a stationary points question.

If you cant do part (a), leave it and come back to it or try working backwards

KEY WORDS : Its likely to be differentiation when you see the phrases

"rate of change" "gradient of tangent"

How can you tell a function is never decreasing?   Find $\frac{dy}{dx}$ and prove it's never negative   (usually by completing the square)