

## Integration

### 8. Integration

Section A - Revision Section - There is no revision section for this topic.

#### Section B - Assessment Standard Section

This section will help you practise for your Assessment Standard Test for Integration 1 (Relationships and Calculus 1.3)

1. Find  $\int \frac{1}{3x^4} dx$ , where  $x \neq 0$ .
2. Find  $\int (4\sqrt{x} + \frac{1}{x^3}) dx$ , where  $x > 0$ .
3.  $f'(x) = 6x^2 - 2x$ , find  $f(x)$ .
4.  $g'(x) = (2x + 3)^{-2}$ , find  $g(x)$ ,  $x \neq \frac{-3}{2}$ .
5. Find  $\int 5\sin\theta d\theta$ .
6. Find  $\int 8\cos\theta d\theta$ .
7. Find the value of  $\int_1^2 (3x^2 + 4) dx$ .
8. Find the value of  $\int_1^2 (4 - x^2) dx$ .

# Integration

## Section C - Operational Skills Section

This section provides problems with the operational skills associated with Exponentials and Logs.

### O1 *I can integrate an algebraic function*

#### 1. Carry out the following integrations

(a)  $\int \sqrt{x} \, dx$

(b)  $\int \sqrt{x^3} \, dx$

(c)  $\int \sqrt{x^5} \, dx$

(d)  $\int \sqrt[3]{x} \, dx$

(e)  $\int \sqrt[3]{x^2} \, dx$

(f)  $\int \sqrt[5]{x^4} \, dx$

(g)  $\int \frac{1}{\sqrt{x}} \, dx$

(h)  $\int \frac{1}{\sqrt[3]{x^2}} \, dx$

(i)  $\int \frac{1}{\sqrt[3]{x^4}} \, dx$

(j)  $\int \frac{1}{\sqrt[5]{x^3}} \, dx$

(k)  $\int \frac{2}{\sqrt[3]{x^8}} \, dx$

(l)  $\int \frac{3}{\sqrt[4]{x^3}} \, dx$

(m)  $\int \frac{1}{2\sqrt[3]{x^2}} \, dx$

(n)  $\int \frac{2}{3\sqrt[4]{x^3}} \, dx$

(o)  $\int \frac{3}{5\sqrt[3]{x^7}} \, dx$

#### 2. Carry out the following integrations

(a)  $\int x^3 + 3x^2 + 5x \, dx$

(b)  $\int 3x^5 + 2x^4 - x \, dx$

(c)  $\int x^2 + 6x - 1 \, dx$

(d)  $\int x^{\frac{2}{3}} + 4x^2 \, dx$

(e)  $\int 3x^{\frac{1}{2}} - 2x^{-5} \, dx$

(f)  $\int 5x^{-2} - 3x^{\frac{1}{2}} \, dx$

(g)  $\int \frac{1}{2\sqrt[3]{x}} + x^2 \, dx$

(h)  $\int 3x^7 - \frac{1}{5\sqrt[4]{x^3}} \, dx$

(i)  $\int \frac{3}{5\sqrt[2]{x^5}} + 5 \, dx$

(j)  $\int \frac{2}{3\sqrt[4]{x^3}} + 2x^2 + x \, dx$

(k)  $\int 5x^2 - \frac{1}{3\sqrt{x^2}} \, dx$

(l)  $\int 4x^{-2} - 4x^{\frac{2}{3}} \, dx$

(m)  $\int 5x^3 - 6x^{-\frac{1}{2}} \, dx$

(n)  $\int 4x^2 + \frac{6}{\sqrt[3]{x}} \, dx$

(o)  $\int x^2 - 5 - \frac{1}{x^2} \, dx$

# Integration

**O2 I can apply a standard integral of the form  $f(x) = (px + q)^n$  with  $n \neq -1$ .**

1. Find

- |                                      |                                      |                                      |
|--------------------------------------|--------------------------------------|--------------------------------------|
| (a) $\int (x + 2)^8 dx$              | (b) $\int (2x + 4)^3 dx$             | (c) $\int (5x + 7)^4 dx$             |
| (d) $\int (2x - 1)^5 dx$             | (e) $\int 6(5 - 4x)^6 dx$            | (f) $\int (10 - x)^{-10} dx$         |
| (g) $\int 3(4x + 1)^{-3} dx$         | (h) $\int 2(5x - 9)^{-5} dx$         | (i) $\int (3 - 7x)^{-4} dx$          |
| (j) $\int (x - 1)^{\frac{1}{2}} dx$  | (k) $\int (2x - 1)^{\frac{1}{3}} dx$ | (l) $\int (2x - 1)^{\frac{1}{4}} dx$ |
| (m) $\int (2x - 2)^{\frac{1}{2}} dx$ | (n) $\int (3x + 4)^{\frac{2}{3}} dx$ | (o) $\int (7 + 3x)^{\frac{3}{4}} dx$ |

2. Find

- |                                  |                                  |  |
|----------------------------------|----------------------------------|--|
| (a) $\int \frac{1}{(5x+3)^5} dx$ | (b) $\int \frac{dx}{(3x-2)^4}$   | (c) $\int \frac{3}{(4-2x)^6} dx$             |
| (d) $\int \frac{2 dx}{(x-2)^3}$  | (e) $\int \frac{3 dx}{(4x+2)^4}$ | (f) $\int \frac{1}{(5x-2)^{\frac{1}{2}}} dx$ |

3. Find

- |                                       |                                    |                                      |
|---------------------------------------|------------------------------------|--------------------------------------|
| (a) $\int \sqrt{4x + 2} dx$           | (b) $\int 6\sqrt{3x + 1} dx$       | (c) $\int \sqrt{9 - 5x} dx$          |
| (d) $\int \sqrt[3]{2x - 3} dx$        | (e) $\int \sqrt[3]{6x - 2} dx$     | (f) $\int \sqrt[4]{2x + 4} dx$       |
| (g) $\int \frac{1}{\sqrt{(3x-4)}} dx$ | (h) $\int \frac{dx}{\sqrt{(x+8)}}$ | (i) $\int \frac{2dx}{\sqrt{(2x-5)}}$ |

**O3 I can integrate  $\sin^2 x$  and  $\cos^2 x$  by first making a substitution.**

Find

- |                        |                       |                        |
|------------------------|-----------------------|------------------------|
| 1. $\int \sin^2 x dx$  | 2. $\int \cos^2 x dx$ | 3. $\int 2\sin^2 x dx$ |
| 4. $\int 2\cos^2 x dx$ |                       |                        |

# Integration

## 04 I can evaluate the definite integral of a function.

1. Find

(a)  $\int_0^1 (x^2 - 3x + 4) dx$

(b)  $\int_0^1 (4x^2 + 3x) dx$

(c)  $\int_0^1 (x^3 + 2x^2 - 1) dx$

(d)  $\int_0^2 (2x - 1)(x + 2) dx$

(e)  $\int_{-1}^1 2x^2 (2x + 1) dx$

(f)  $\int_{-2}^1 (2x^3 - x^2 + 3x) dx$

2. Find

(a)  $\int_{-1}^1 (5x^3 - 2x) dx$

(b)  $\int_{-1}^1 (3x^2 - 4x + 2) dx$

(c)  $\int_{-1}^1 (3x + 2)(x - 2) dx$

(d)  $\int_0^2 (3x^2 + 8x - 5) dx$

(e)  $\int_{-2}^0 (x - 3)^2 dx$

(f)  $\int_{-1}^0 (x^2 - 2x + 7) dx$

(g)  $\int_0^3 x(x - 2)(x - 3) dx$

(h)  $\int_{-2}^2 (x + 2)(x - 2) dx$

(i)  $\int_1^4 (x - 1)(x - 2) dx$

3. Evaluate

(a)  $\int_0^\pi \cos 2x dx$

(b)  $\int_0^{\pi/2} \cos 2x dx$

(c)  $\int_0^\pi \sin 2x dx$

(d)  $\int_0^{\pi/4} \sin 2x dx$

(e)  $\int_0^{\pi/3} \cos 3x dx$

(f)  $\int_0^{2\pi} \cos \frac{1}{2}x dx$

4. Evaluate

(a)  $\int_0^\pi (\sin t + \cos t) dt$

(b)  $\int_0^{\pi/4} \sin 4t + \cos 4t dt$

(c)  $\int_0^{\pi/4} \cos \left( 2t + \frac{\pi}{2} \right) dt$

(d)  $\int_{\pi/6}^{\pi/4} \sin \left( 2t - \frac{\pi}{3} \right) dt$

# Integration

5. Evaluate

(a)  $\int_0^{1/2} (x^3 + 12x^2 + 7) dx$

(b)  $\int_{-1}^{1/2} (3x^2 - 4x) dx$

(c)  $\int_0^{2/3} (9x^2 + 8) dx$

(d)  $\int_{-1/2}^1 (9x^2 + 2x - 1) dx$

(e)  $\int_0^{\sqrt{3}} (2x + 4) dx$

(f)  $\int_1^{\sqrt{3}} (10 - 2x) dx$

**05 I can evaluate one of the limits of a definite integral given the value of the definite integral.**

1. Find  $a$ , when  $a > 0$

(a)  $\int_0^a (2x + 2) dx = 8$

(b)  $\int_0^a x^2 dx = \frac{64}{3}$

2. Given that,  $\int_0^a 3x^{1/2} dx = 16$ , calculate the value of  $a$ .

3. Find  $a$  for  $0 \leq t \leq 2\pi$  given:

(a)  $\int_0^a \cos t dt = 1$

(b)  $\int_0^a \sin t dt = 2$

4. Given that  $\int_0^a 5\sin 3x dx = \frac{10}{3}$ ,  $0 \leq a \leq \pi$ , calculate the value of  $a$ .

5. Determine  $p$ , given that  $\int_1^p x^{1/2} dx = 42$

6. Given that  $\int_0^k \frac{1}{(4-3x)^2} dx = \frac{1}{36}$ , find  $k$ .

# Integration

**06** I can solve differential equations of the form  $\frac{dy}{dx} = f(x)$  and give a particular solution.

- Given the gradient  $\frac{dy}{dx}$  of the curve at the point  $(x, y)$  and a point on the curve, find the equation of each curve:
  - $\frac{dy}{dx} = 3x^2 - 6x + 1$  (3,4)
  - $\frac{dy}{dx} = 4x^3 - 6x^2$  (1,9)
- Find the solution to the following differential equations:
  - $\frac{dy}{dx} = 4x^3 + \frac{2}{x^3}$  and  $y = 0$  when  $x = 1$
  - $\frac{dy}{du} = \frac{u^2+1}{u^2}$  and  $y = 4$  when  $u = 2$
- A curve has gradient given by  $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ . The curve passes through the point (9,10). Find the equation of the curve.
- The graph of  $y = f(x)$  passes through the point  $(\frac{\pi}{9}, 1)$ .  
If  $f'(x) = \sin 3x$ , express  $y$  in terms of  $x$ .
- A curve for which  $\frac{dy}{dx} = 3 \sin 2x$  passes through the point  $(\frac{5}{12}\pi, \sqrt{3})$ .
- A point moves in a straight line such that its acceleration  $a$  is given by  $a = 2(4 - t)^{\frac{1}{2}}$ ,  $0 \leq t \leq 4$ . If it starts at rest, find an expression for the velocity  $v$  where  $a = \frac{dv}{dt}$ .
- The curve  $y = f(x)$  is such that  $\frac{dy}{dx} = 4x - 6x^2$ . The curve passes through the point  $(-1, 9)$ . Express  $y$  in terms of  $x$ .

# Integration

## Cross Topic Questions

### Integration and the wave function

1. (a) The expression  $3 \sin x - 5 \cos x$  can be written in the form  $R \sin(x + a)$  where  $R > 0$  and  $0 \leq a \leq 2\pi$ .

Calculate the values of  $R$  and  $a$ .

- (b) Hence find the value of  $t$ , where  $0 \leq t \leq 2$ , for which

$$\int_0^t (3 \cos x + 5 \sin x) dx = 3$$

### Integration and Functions

2. (a) The functions  $f$  and  $g$  are defined as  $f(x) = 3x + 2$  and  $g(x) = x^5$ .

Find  $g(f(x))$ .

- (b) If  $p'(x) = g(f(x))$ , and  $p(-\frac{1}{3}) = 1$ , find  $p(x)$ .

### Integration and Rates of Change

3. The rate of change of the temperature,  $T^\circ\text{C}$ , of a mug of coffee is given by

$$\frac{dT}{dt} = \frac{1}{25}t - k, 0 \leq t \leq 50$$

With  $t$ , the time elapsed in minutes since pouring and  $k$  a constant.

Initially the temperature of the coffee is  $100^\circ\text{C}$ .

After 10 minutes it is  $82^\circ\text{C}$ .

Express  $T$  in terms of  $t$ .

# Integration

## Answers

### Section B

1.  $-\frac{1}{9}x^{-3} + c$       2.  $\frac{8}{3}x^{\frac{1}{3}} - \frac{1}{2}x^{-2} + c$   
3.  $f(x) = 2x^3 - x^2 + c$       4.  $g(x) = \frac{(2x+3)^{-1}}{-2} + c$       5.  $-5\cos\theta + c$   
6.  $8\sin\theta + c$       7. 11 units      8.  $1\frac{2}{3}$  units.

### Section C

#### 01

1. (a)  $\frac{2}{3}x^{\frac{3}{2}} + c$       (b)  $\frac{2}{5}x^{\frac{5}{2}} + c$       (c)  $\frac{2}{7}x^{\frac{7}{2}} + c$   
(d)  $\frac{3}{4}x^{\frac{4}{3}} + c$       (e)  $\frac{3}{5}x^{\frac{5}{3}} + c$       (f)  $\frac{5}{9}x^{\frac{9}{5}} + c$   
(g)  $2x^{\frac{1}{2}} + c$       (h)  $3x^{\frac{1}{3}} + c$       (i)  $-\frac{3}{x^3} + c$   
(j)  $\frac{5}{2}x^{\frac{2}{5}} + c$       (k)  $-\frac{6}{5x^{\frac{5}{3}}} + c$       (l)  $12x^{\frac{1}{4}} + c$   
(m)  $\frac{3}{2}x^{\frac{1}{3}} + c$       (n)  $\frac{8}{3}x^{\frac{1}{4}} + c$       (o)  $-\frac{9}{20x^{\frac{4}{3}}} + c$
2. (a)  $\frac{x^4}{4} + x^3 + \frac{5}{2}x^2 + c$       (b)  $\frac{x^6}{2} + \frac{2}{5}x^5 + \frac{x^2}{2} + c$   
(c)  $\frac{x^3}{3} + 3x^2 - x + c$       (d)  $\frac{3}{5}x^{\frac{5}{3}} + \frac{4}{3}x^3 + c$   
(e)  $2x^{\frac{3}{2}} + \frac{1}{2x^4} + c$       (f)  $-\frac{5}{x} - 2x^{\frac{3}{2}} + c$   
(g)  $\frac{3}{4}x^{\frac{2}{3}} + \frac{1}{3}x^3 + c$       (h)  $\frac{3}{8}x^8 - \frac{4}{5}x^{\frac{1}{4}} + c$   
(i)  $-\frac{6}{15\sqrt{x^3}} + 5x + c$       (j)  $\frac{8}{3}x^{\frac{1}{4}} + \frac{2}{3}x^3 + \frac{1}{2}x^2 + c$   
(k)  $\frac{5}{3}x^3 - 3x^{\frac{1}{3}} + c$       (l)  $-\frac{4}{x} - \frac{12}{5}x^{\frac{5}{3}} + c$   
(m)  $\frac{5}{4}x^4 - 12x^{\frac{1}{2}} + c$       (n)  $\frac{4}{3}x^3 + 9x^{\frac{2}{3}} + c$   
(o)  $\frac{x^3}{3} - 5x - \frac{1}{x} + c$

# Integration

02

1. (a)  $\frac{1}{9}(x+9)^9 + c$       (b)  $\frac{1}{8}(2x+4)^4 + c$       (c)  $\frac{1}{25}(5x+7)^5 + c$   
(d)  $\frac{1}{12}(2x-1)^6 + c$       (e)  $-\frac{3}{14}(5-4x)^7 + c$       (f)  $\frac{1}{9}(10-x)^{-9} + c$   
(g)  $-\frac{3}{8}(4x+1)^{-2} + c$       (h)  $-\frac{1}{10}(5x-9)^{-4} + c$       (i)  $\frac{1}{21}(3-7x)^{-3} + c$   
(j)  $\frac{2}{3}(x-1)^{\frac{3}{2}} + c$       (k)  $\frac{3}{8}(2x-1)^{\frac{4}{3}} + c$       (l)  $\frac{2}{5}(2x-1)^{\frac{5}{4}} + c$   
(m)  $\frac{1}{3}(2x-2)^{\frac{3}{2}} + c$       (n)  $\frac{1}{5}(3x+4)^{\frac{5}{3}} + c$       (o)  $\frac{4}{21}(7+3x)^{\frac{7}{4}} + c$
2. (a)  $-\frac{1}{20}(5x+3)^{-4} + c$       (b)  $-\frac{1}{9}(3x-2)^{-3} + c$       (c)  $\frac{3}{10}(4-2x)^{-5} + c$   
(d)  $-(x-2)^{-2} + c$       (e)  $-\frac{1}{4}(4x+2)^{-3} + c$       (f)  $\frac{2}{5}(5x-2)^{\frac{1}{2}} + c$
3. (a)  $\frac{1}{6}(4x+2)^{\frac{3}{2}} + c$       (b)  $\frac{4}{3}(3x+1)^{\frac{3}{2}} + c$       (c)  $-\frac{2}{15}(9-5x)^{\frac{3}{2}} + c$   
(d)  $\frac{3}{8}(2x-3)^{\frac{4}{3}} + c$       (e)  $\frac{1}{8}(6x-2)^{\frac{4}{3}} + c$       (f)  $\frac{2}{5}(2x+4)^{\frac{5}{4}} + c$   
(g)  $\frac{2}{3}(3x-4)^{\frac{1}{2}} + c$       (h)  $2(x+8)^{\frac{1}{2}} + c$       (i)  $2(2x-5)^{\frac{1}{2}} + c$

03

1.  $\frac{1}{2}x - \frac{1}{4}\sin 2x + c$       2.  $\frac{1}{2}x + \frac{1}{4}\sin 2x + c$   
3.  $x - \frac{1}{2}\sin 2x + c$       4.  $x + \frac{1}{2}\sin 2x + c$

04

1. (a)  $\frac{17}{6}$       (b)  $\frac{17}{6}$       (c)  $-\frac{1}{12}$       (d)  $\frac{22}{3}$       (e)  $\frac{4}{3}$       (f)  $-3$   
2. (a)  $0$       (b)  $6$       (c)  $-6$       (d)  $4$       (e)  $\frac{98}{3}$       (f)  $\frac{25}{3}$   
(g)  $\frac{9}{4}$       (h)  $-\frac{32}{3}$       (i)  $\frac{9}{2}$   
3. (a)  $0$       (b)  $0$       (c)  $0$       (d)  $\frac{1}{2}$       (e)  $0$       (f)  $0$   
4. (a)  $2$       (b)  $\frac{1}{2}$       (c)  $-\frac{1}{2}$       (d)  $\frac{2-\sqrt{3}}{4}$

# Integration

5. (a)  $\frac{257}{64}$  (b)  $\frac{21}{8}$  (c)  $\frac{56}{9}$  (d)  $\frac{21}{8}$  (e)  $3+4\sqrt{3}$   
(f)  $10\sqrt{3} - 12$

## 05

- 1(a)  $a = -4, a = 2$  1(b)  $a = 4$  2.  $a = 4$  3(a)  $a = \frac{\pi}{2}$   
3(b)  $a = \pi$  4.  $a = \frac{\pi}{3}, \pi$  5.  $p = 16$  6.  $k = \frac{1}{3}$

## 06

- 1(a)  $y = x^3 - 3x^2 + x + 1$  1(b)  $y = x^4 - 2x^3 + 10$   
2(a)  $y = x^4 - \frac{1}{x^2}$  2(b)  $y = u - \frac{1}{u} + \frac{5}{2}$  3.  $y = 2\sqrt{x} + 4$   
4.  $f(x) = \frac{7}{6} - \frac{1}{3}\cos 3x$  5.  $y = \frac{\sqrt{3}}{4} - \frac{3}{2}\cos 2x$   
6.  $V(t) = \frac{32}{3} - \frac{4}{3}(4 - t)^{\frac{3}{2}}$  7.  $y = 2x^2 - 2x^3 + 5$

## Section D

1. (a)  $\sqrt{34}\sin(x + 5 \cdot 25)$  (b)  $t = 0 \cdot 68$  radians  
2. (a)  $(3x + 2)^5$  (b)  $p(x) = \frac{(3x+2)^6}{18} + \frac{17}{18}$   
3.  $T = \frac{1}{50}t^2 - 2t + 100$