EF4. Vectors

Section A - Revision Section

This section will help you revise previous learning which is required in this topic.

R1 I have revised National 5 vectors and 3D coordinate.

1. If vector $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and vector $\boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, find the resultant vector of:

(a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $\boldsymbol{a} - \boldsymbol{b}$ (c) $3\boldsymbol{a} + \boldsymbol{b}$

(d) $\boldsymbol{a} - 2\boldsymbol{b}$ (e) $5\boldsymbol{a} - 3\boldsymbol{b}$ (f) $2\boldsymbol{a} + 4\boldsymbol{b}$

2. If vector $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and vector $\boldsymbol{b} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, find the resultant vector of

(a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $\boldsymbol{a} - \boldsymbol{b}$ (c) $2\boldsymbol{a} + 3\boldsymbol{b}$

(d) $5\boldsymbol{a} - \boldsymbol{b}$ (e) $3\boldsymbol{a} - 2\boldsymbol{b}$ (f) $\boldsymbol{a} + 4\boldsymbol{b}$

3. If $\boldsymbol{p} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$, find:

(a) $|\boldsymbol{p}|$ (b) $|\boldsymbol{q}|$ (c) $|\boldsymbol{p} + \boldsymbol{q}|$

(d) $|\boldsymbol{p} - \boldsymbol{q}|$ (e) $|3\boldsymbol{p} - \boldsymbol{q}|$ (f) $|2\boldsymbol{p} + 3\boldsymbol{q}|$

4. Three vectors are defined as $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$ and $\overrightarrow{EF} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$, find:

(a) $|\overrightarrow{AB}|$ (b) $|\overrightarrow{CD}|$ (c) $|\overrightarrow{EF}|$

Vectors

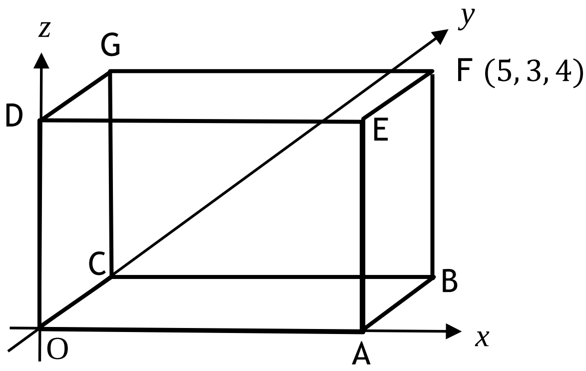
5. Three points A, B and C have the coordinates $(2, 5, 3)$, $(-1, 3, 0)$ and $(1, 4, 2)$ respectively. Find the vectors

(a) $\overrightarrow{OA}$ **(b)** $\overrightarrow{OB}$ **(c)** $\overrightarrow{OC}$

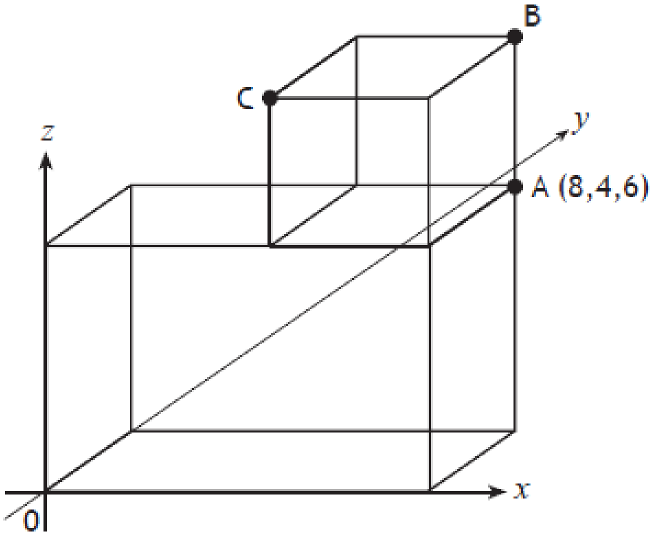
(d) $\overrightarrow{AB}$ **(e)** $\overrightarrow{BC}$ **(f)** $\overrightarrow{AC}$

6. The diagram shows the cuboid OABCDEFG. O is the origin and OA, OC and OD are aligned with the x, y and z axes respectively. The point F has coordinates $(5, 3, 4)$.

List the coordinates of the other six vertices.



7.

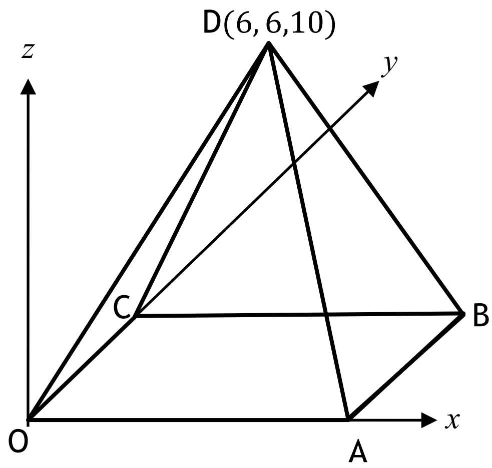


The diagram shows a cube placed on top of a cuboid, relative to the coordinate axes. A is the point $(8, 4, 6)$.

Write down the coordinates of B and C.

8. The diagram shows the square based pyramid DOABC. O is the origin with OA and OC aligned with the x and y axes respectively. The point D has coordinates $(6, 6, 10)$.

Write down the coordinates of the points A, B and C.

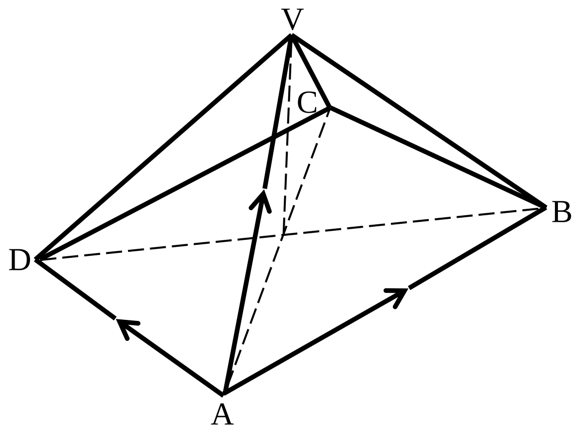


Vectors

Section B - Assessment Standard Section

This section will help you practise for your Assessment Standard Test (Expressions and Functions 1.4)

1. VABCD is a pyramid with rectangular base ABCD.



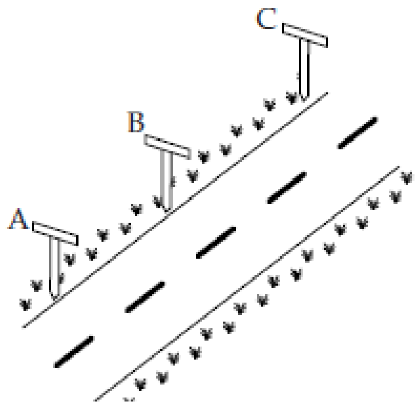
The vectors $\overrightarrow{AB}, \overrightarrow{AD}$ and $\overrightarrow{AV}$ are given by

$$\overrightarrow{AB} = \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix}; \quad \overrightarrow{AD} = \begin{pmatrix} -2 \\ 10 \\ -2 \end{pmatrix} \text{ and } \overrightarrow{AV} = \begin{pmatrix} 1 \\ 7 \\ 7 \end{pmatrix}.$$

Express $\overrightarrow{CV}$ in component form.

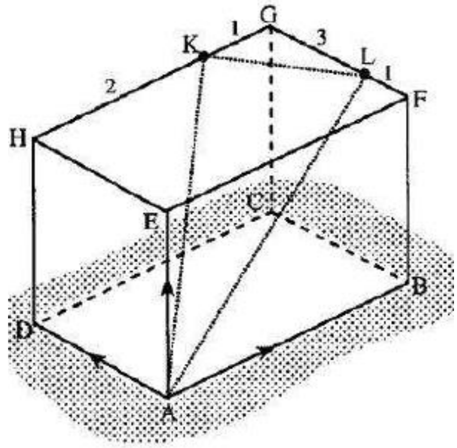
2. Road makers look along the tops of a set of T-rods to ensure that straight sections of road are being created.

Relative to suitable axes the top left corners of the T-rods are the points A (-8, -10, -2), B (-2, -1, 1) and C(6, 11, 5).



Determine whether or not the section of road ABC has been built in a straight line.

Vectors



3. ABCDEFGH is a cuboid.

K lies two thirds of the way along HG.
(i.e. HK:KG = 2:1).

L Lies one quarter of the way along FG.
(i.e. FL:LG = 1:3).

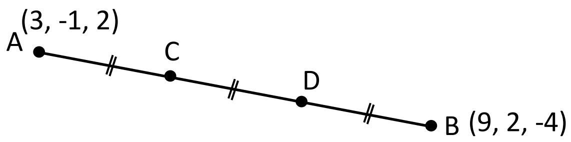
$\overrightarrow{AB}, \overrightarrow{AD}$ and $\overrightarrow{AE}$ can be represented by the vectors

$\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ respectively.

(a) Calculate the components of $\overrightarrow{AK}$.

(b) Calculate the components of $\overrightarrow{AL}$.

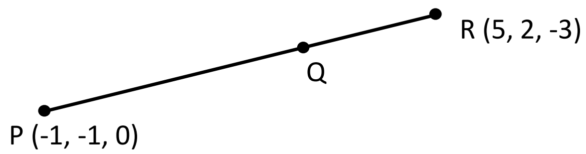
4. The line AB is divided into 3 equal parts by the points C and D, as shown.
A and B have coordinates (3, -1, 2) and (9, 2, -4).



(a) Find the components of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Find the coordinates of C and D.

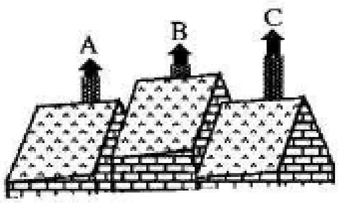
5. The point Q divides the line joining P (-1, -1, 0) to R (5, 2, -3) in the ratio 2:1.



Find the coordinates of Q.

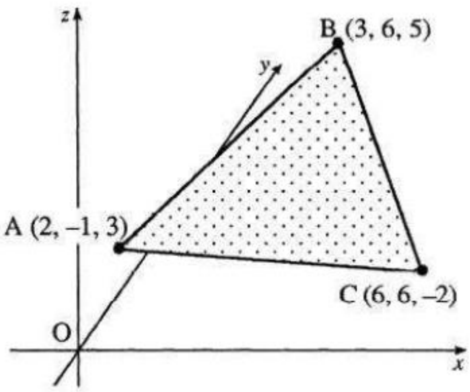
Vectors

6. Relative to a suitable set of axes, the tops of three chimneys have coordinates given by A(1, 3, 2), B (2, -1, 4) and C (4, -9, 8).



Show that A, B and C are collinear.

7. A triangle ABC has vertices A (2, -1, 3), B (3, 6, 5) and C (6, 6, -2).



(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Calculate the size of angle BAC.

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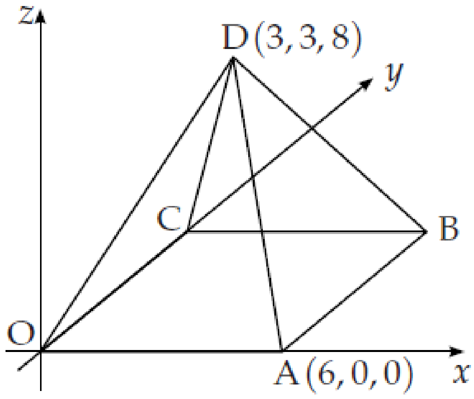
8. The diagram shows a square-based pyramid of height 8 units.

Square OABC has a side length of 6 units.

The coordinates of A and D are (6, 0, 0) and (3, 3, 8).

C lies on the y-axis.



(a) Write down the coordinates of B.

(b) Determine the components of $\overrightarrow{DA}$ and $\overrightarrow{DB}$.

(c) Calculate the size of angle ADB.

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Section C – Operational Skills Section

This section provides problems with the operational skills associated with Exponentials and Logs

O1	***I can express and manipulate vectors in the form $\boldsymbol{ai} + \boldsymbol{bj} + \boldsymbol{ck}$.***

1. **Write the following vectors, given in unit vector form, in component form.**

(a) $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ (b) $\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{j}$ (c) $\boldsymbol{c} = \boldsymbol{i} - 6\boldsymbol{j} - \boldsymbol{4k}$

2. **Write the following vectors, given in component form, in unit vector form.**

(a) $\boldsymbol{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (b) $\boldsymbol{q} = \begin{pmatrix} 6 \\ -2 \\ 7 \end{pmatrix}$ (c) $\boldsymbol{r} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}$

3. Two vectors are defined, in unit vector form, as $\boldsymbol{p} = 3\boldsymbol{i} - \boldsymbol{k}$ and $\boldsymbol{q} = \boldsymbol{i} - \boldsymbol{2j} + \boldsymbol{3k}$.

(a) Express $\boldsymbol{p} + 2\boldsymbol{q}$ **in unit vector form.**

(b) Express $3\boldsymbol{p} - 4\boldsymbol{q}$ **in unit vector form.**

(c) Find $|\boldsymbol{p} + 2\boldsymbol{q}\,|$.

(d) Find $|3\boldsymbol{p} - 4\boldsymbol{q}|$.

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O2	***I can calculate the scalar product and know that perpendicular vectors have a scalar product of zero.***

1. Find the scalar product of each of the pairs of vectors below and state clearly which pairs are perpendicular.

(a) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$.

(b) $\boldsymbol{p} = \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$.

(c) $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{4j} + \boldsymbol{2k}$ and $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{3j} + \boldsymbol{k}$

2. If $|\overrightarrow{AB}| = 3$ and $|\overrightarrow{AC}| = 4$ and $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are inclined at an angle of $60°$, find the scalar product $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

3. If $|\boldsymbol{a}| = \frac{\sqrt{2}}{3}$ and $|\boldsymbol{b}| = \frac{3}{4}$ and $\boldsymbol{a}$ and $\boldsymbol{b}$ are inclined at an angle of $45°$, find the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$.

O3	***I can determine whether or not coordinates are collinear, using the appropriate language, and can apply my knowledge of vectors to divide lines in a given ratio.***

1. The point Q divides the line joining P$(-1, -1, 3)$ and R$(5, -1, -3)$ in the ratio 5:1. Find the coordinates of Q.

2. The point B divides the line joining A$(1, -2, 4)$ and C$(-11, 13, -8)$ in the ratio 1:2. Find the coordinates of B.

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3. John is producing a 3D design on his computer.

Relative to suitable axes 3 points in his design have coordinates P$(-3, 4, 7)$, Q$(-1, 8, 3)$ and R$(0, 10, 1)$.

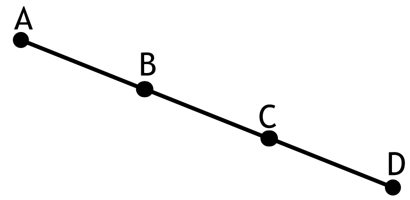
(a) Show that P, Q and R are collinear.

(b) Find the coordinates of S such that $\overrightarrow{PS} = 4\overrightarrow{PQ}$.

4. A and B are the points $(0, -2, 3)$ and $(3, 0, 2)$ respectively.

B and C are the points of trisection of AD, that is AB = BC = CD.

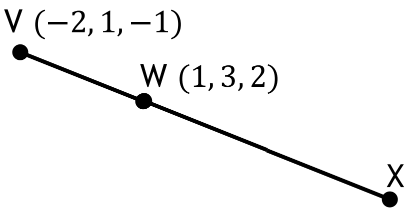
Find the coordinates of D.



5. The points V, W and X are shown on the line opposite.

V, W and X are collinear points such that WX = 2VW.

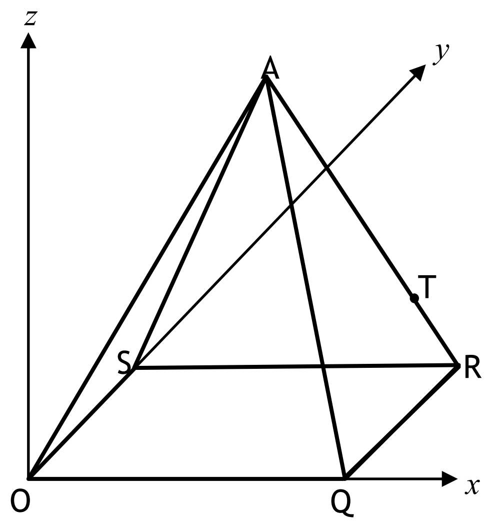
Find the coordinates of X.



6. AOQRS is a pyramid. Q is the point $(16, 0, 0)$, R is $(16, 8, 0)$ and A is $(8, 4, 12)$. T divides RA in the ratio 1:3.

(a) Find the coordinates of the point T.

(b) Express $\overrightarrow{QT}$ in component form.



Vectors

O4 I can apply knowledge of vectors to find an angle in three dimensions.

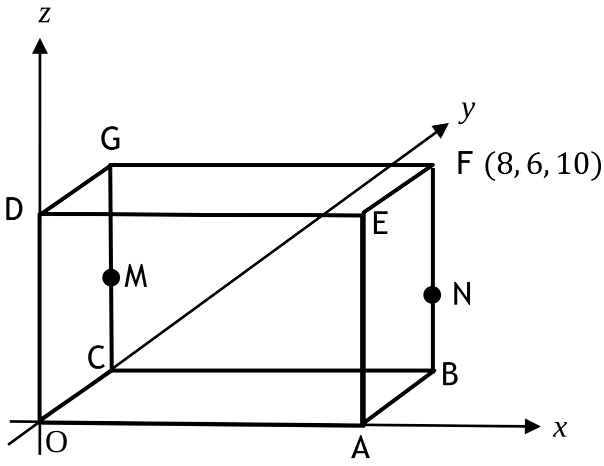
1. Three planes, Tango (T), Delta (D) and Bravo (B) are being tracked by radar. Relative to a suitable origin, the positions of the three planes are T$(23, 0, 8)$, D$(-12, 0, 9)$ and B$(28, -15, 7)$

(a) Express the vectors $\overrightarrow{BT}$ and $\overrightarrow{BD}$ in component form.

(b) Find the size of angle TBD.

2. The diagram shows a cuboid OABCDEFG with the lines OA, OC and OD lying on the axes.

The point F has coordinates $(8, 6, 10)$, M is the midpoint of CG and N divides BF in the ratio 2:3.

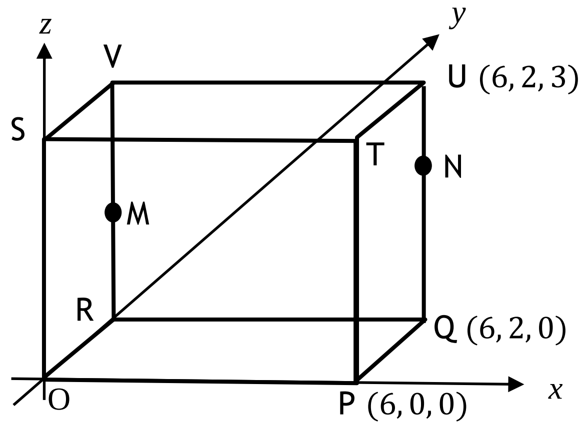


(a) State the coordinates of A, M and N.

(b) Determine the components of the vectors $\overrightarrow{MA}$ and $\overrightarrow{MN}$.

(c) Find the size of angle AMN.

3. In the diagram OPQRSTUV is a cuboid. M is the midpoint of VR and N is the point on UQ such that $\text{UN} = \frac{1}{3}\text{UQ}$.



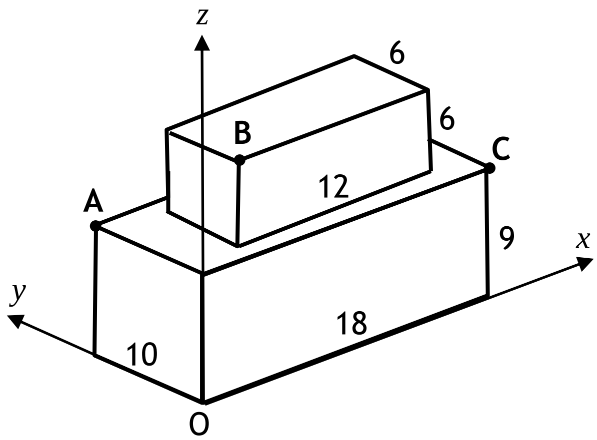
(a) State the coordinates of T, M and N.

(b) Determine the components of the vectors $\overrightarrow{TM}$ and $\overrightarrow{TN}$.

(c) Find the size of angle MTN.

4. A cuboid measuring 12cm by 6cm by 6cm is placed centrally on top of another cuboid measuring 18cm by 10cm by 9cm.

Coordinate axes are taken as shown.



(a) The point A has coordinates $(0, 10, 9)$ and the point C has coordinates $(18, 0, 9)$. Write down the coordinates of B.

(b) Find the size of angle ABC.

O5 I know the properties of the scalar product and their uses.

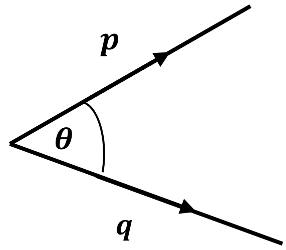
1. Vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are defined by $\boldsymbol{p} = -3\boldsymbol{i} - 12\boldsymbol{k}$ and $\boldsymbol{q} = 8\boldsymbol{i} + 7\boldsymbol{j} - 2\boldsymbol{k}$.

Determine whether or not $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular to each other.

2. For what value of p are the vectors $\boldsymbol{a} = \begin{pmatrix} p \\ -2 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 14 \\ 2p \end{pmatrix}$ perpendicular?

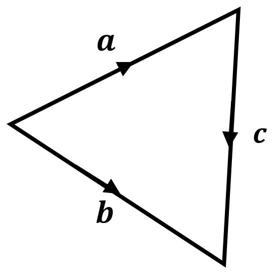
3. The diagram shows vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

If $|\boldsymbol{p}| = 3$, $|\boldsymbol{q}| = 4$ and $\boldsymbol{p}.(\boldsymbol{p} + \boldsymbol{q}) = 15$, find the size of the acute angle θ between $\boldsymbol{p}$ and $\boldsymbol{q}$.



Vectors

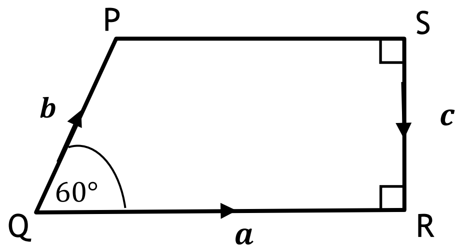
4. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ form an equilateral triangle of length 3 units.



(a) Find the scalar product $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c})$.

(b) What does this tells us about the vectors $\boldsymbol{a}$ and $\boldsymbol{b}+\boldsymbol{c}$.

5. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are shown on the diagram. Angle PQR = $60°$.



It is also given that $|\boldsymbol{a}| = 3$ and $|\boldsymbol{b}| = 2$.

(a) **Evaluate $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c})$ and $\boldsymbol{c}.(\boldsymbol{a}-\boldsymbol{b})$.**

(b) **Find $|\boldsymbol{b}+\boldsymbol{c}|$ and $|\boldsymbol{a}-\boldsymbol{b}|$.**

Vectors

Answers

Section A

R1

1. **(a)** $\begin{pmatrix}5\\5\end{pmatrix}$ **(b)** $\begin{pmatrix}-1\\-3\end{pmatrix}$ **(c)** $\begin{pmatrix}9\\7\end{pmatrix}$ **(d)** $\begin{pmatrix}-4\\-7\end{pmatrix}$ **(e)** $\begin{pmatrix}1\\-7\end{pmatrix}$ **(f)** $\begin{pmatrix}16\\18\end{pmatrix}$

2. **(a)** $\begin{pmatrix}5\\4\\3\end{pmatrix}$ **(b)** $\begin{pmatrix}1\\-4\\-1\end{pmatrix}$ **(c)** $\begin{pmatrix}12\\12\\8\end{pmatrix}$ **(d)** $\begin{pmatrix}13\\-4\\3\end{pmatrix}$ **(e)** $\begin{pmatrix}5\\-8\\-1\end{pmatrix}$ **(f)** $\begin{pmatrix}11\\16\\9\end{pmatrix}$

3. **(a)** $\sqrt{14}$ **(b)** $\sqrt{10}$ **(c)** $\sqrt{26}$ **(d)** $\sqrt{22}$ **(e)** $\sqrt{130}$ **(f)** $\sqrt{158}$

4. **(a)** $\sqrt{13}$ **(b)** 3 **(c)** $\sqrt{27}$

5. **(a)** $\begin{pmatrix}2\\5\\3\end{pmatrix}$ **(b)** $\begin{pmatrix}-1\\3\\0\end{pmatrix}$ **(c)** $\begin{pmatrix}1\\4\\2\end{pmatrix}$ **(d)** $\begin{pmatrix}-3\\-2\\-3\end{pmatrix}$ **(e)** $\begin{pmatrix}2\\1\\2\end{pmatrix}$ **(f)** $\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$

6. $A(5,0,0)$, $B(5,3,0)$, $C(0,3,0)$, $D(0,0,4)$, $E(5,0,4)$, $G(0,3,4)$

7. $B(8,4,10)$, $C(4,0,10)$.

8. $A(12,0,0)$, $B(12,12,0)$, $C(0,12,0)$.

Section B

1. $\overrightarrow{CV} = \begin{pmatrix}-5\\-5\\7\end{pmatrix}$

2. Since $7\overrightarrow{AB} = 3\overrightarrow{AC}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel and since A is a common point, A, B and C are collinear.

3. (a) $\overrightarrow{AK} = \begin{pmatrix}-5\\5\\11\end{pmatrix}$ (b) $\overrightarrow{AL} = \begin{pmatrix}2\\4\\9\end{pmatrix}$

4. (a) $\overrightarrow{AB} = \begin{pmatrix}6\\3\\-6\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}2\\1\\-2\end{pmatrix}$ (b) C (5, 0, 0) and D (7, 1, -2)

5. Q (3, 1, -2)

6. Since $\overrightarrow{AC} = 3\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are parallel and since A is a common point, A, B and C are collinear.

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7. (a) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} 4 \\ 7 \\ -5 \end{pmatrix}$ (b) $B\hat{A}C = 51 \cdot 9°$

8. (a) B(6, 6, 0) (b) $\overrightarrow{DA} = \begin{pmatrix} 3 \\ -3 \\ -8 \end{pmatrix}$ $\overrightarrow{DB} = \begin{pmatrix} 3 \\ 3 \\ -8 \end{pmatrix}$ (c) $A\widehat{D}B = 38 \cdot 7°$

O1

1. **(a)** $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 \\ -6 \\ -4 \end{pmatrix}$

2. **(a)** $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ **(b)** $6\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k}$ **(c)** $\boldsymbol{i} - 4\boldsymbol{j}$

3. **(a)** $5\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ **(b)** $5\boldsymbol{i} + 8\boldsymbol{j} - 15\boldsymbol{k}$ **(c)** $\sqrt{66}$ **(d)** $\sqrt{314}$

O2

1. **(a)** 23 **(b)** 0 (perpendicular) **(c)** -13

2. 6

3. $\frac{1}{4}$

O3

1. $Q(4, -1, -2)$ **2.** $B(-3, 3, 0)$

3. **(a)** $\overrightarrow{QR} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, and $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ with conclusion

   (b) $S(5, 20, -9)$

4. $D(9, 4, 0)$ **5.** $X(7, 7, 8)$ **6. (a)** $T(14, 7, 3)$ **(b)** $\overrightarrow{QT} = \begin{pmatrix} -2 \\ 7 \\ 3 \end{pmatrix}$

O4

1. **(a)** $\overrightarrow{BT} = \begin{pmatrix} -5 \\ 15 \\ 1 \end{pmatrix}$ and $\overrightarrow{BD} = \begin{pmatrix} -40 \\ 15 \\ 2 \end{pmatrix}$ **(b)** $50 \cdot 9°$

2. **(a)** $A(8, 0, 0)$, $M(0, 6, 5)$, $N(8, 6, 4)$ **(b)** $\overrightarrow{MA} = \begin{pmatrix} 8 \\ -6 \\ -5 \end{pmatrix}$ and $\overrightarrow{MN} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix}$

   (c) $40 \cdot 0°$

Vectors

3. **(a)** $T(6, 0, 3)$, $M(0, 2, 1\cdot 5)$, $N(6, 2, 2)$ **(b)** $\overrightarrow{TM} = \begin{pmatrix} -6 \\ 2 \\ -1\cdot 5 \end{pmatrix}$ and $\overrightarrow{TN} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

(c) $67\cdot 8°$

4. **(a)** $B(3, 2, 15)$ **(b)** $98\cdot 5°$

O5

1. $\boldsymbol{p}\cdot\boldsymbol{q} = 0$ therefore $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular.

2. $p = 4$

3. $\theta = 60°$

4. **(a)** $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = 0$ **(b)** $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b} + \boldsymbol{c}$

5. **(a)** $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = 3, \boldsymbol{c}.(\boldsymbol{a} - \boldsymbol{b}) = 3$ **(b)** $|\boldsymbol{b} + \boldsymbol{c}| = 1$, $|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{7}$.