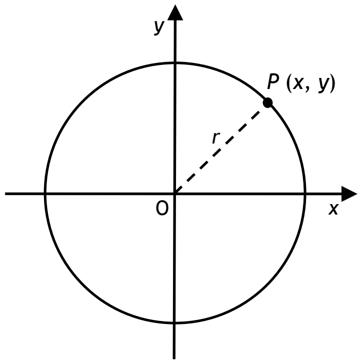
The Circle

If we draw, suitable to relative axes, a circle, radius r, centred on the origin, then the distance from the centre of any point P (x, y) could be determined to be $d=\sqrt{x^2+y^2}$.



As the shape is a circle, then this distance is equal to the radius. It therefore follows that:

Since $r=\sqrt{x^2+y^2}$, then $r^2=x^2+y^2$

Therefore,

The equation $x^2 + y^2 = r^2$ describes a circle with centre (0, 0) and radius r

Example 1: Write down the centre and radius of each circle.

a) $x^2 + y^2 = 64$

b) $x^2 + y^2 = 361$

c) $x^2 + y^2 = \frac{3}{25}$

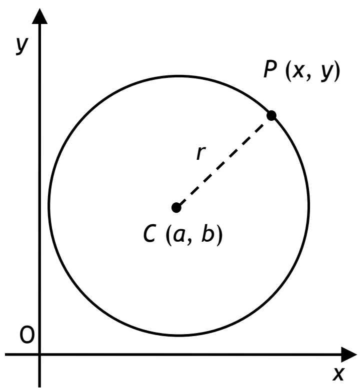
Example 2: State where the points (-2, 7), (6, -8) and (5, 9) lie in relation to the circle $x^2 + y^2 = 100$.

Circles with Centres *Not* at the Origin

The radius in the above circle is the distance between (x, y) and the origin, i.e. $r=\sqrt{(x-0)^2+(y-0)^2}$. If we move the centre to the point (a, b), then $r=\sqrt{(x-a)^2+(y-b)^2}$.

Squaring both sides, we can now also say that:

The equation $(x - a)^2 + (y - b)^2 = r^2$ describes a circle with centre (a, b) and radius r



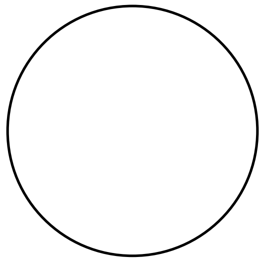
Example 3: Write down the centre and radius of each circle.

a) $(x - 1)^2 + (y + 3)^2 = 4$

b) $(x + 9)^2 + (y - 2)^2 = 20$

c) $(x - 5)^2 + y^2 = 400$

Example 4: A is the point (4, 9) and B is the point (-2, 1).
Find the equation of the circle for which AB is the diameter.



Example 5: Points P, Q and R have coordinates (-10, 2), (5, 7) and (6, 4) respectively.

a) Show that triangle PQR is right angled at Q.

b) Hence find the equation of the circle passing through points P, Q and R.

The General Equation of a Circle

For the circle described in Example 3a, we could expand the brackets and simplify to obtain the equation $x^2 + y^2 - 2x + 6y + 6 = 0$, which would **also** describe a circle with centre (1, -3) and radius 2.

For $x^2 + y^2 + 2gx + 2fy + c = 0$,

$$(x^2 + 2gx) + (y^2 + 2fy) = -c$$

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

$$(x + g)^2 + (y + f)^2 = (g^2 + f^2 - c)$$

Therefore, the circle described by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

has centre $(-g, -f)$ and $r = \sqrt{g^2 + f^2 - c}$

Example 6: Find the centre and radius of the circle with equation $x^2 + y^2 - 4x + 8y - 5 = 0$

Example 7: State why the equation $x^2 + y^2 - 4x - 4y + 15 = 0$ does **not** represent a circle.

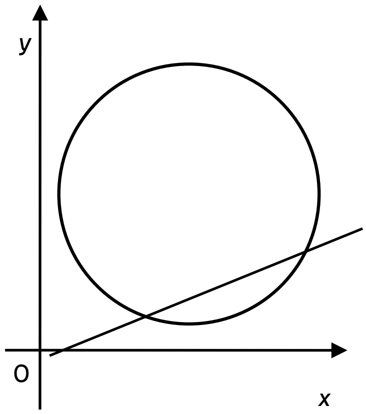
Example 8: State the range of values of c such that the equation $x^2 + y^2 - 4x + 6y + c = 0$ describes a circle.

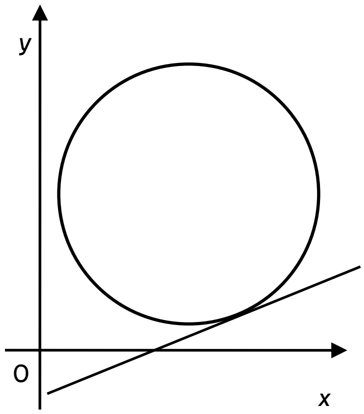
Example 9: Find the equation of the circle concentric with $x^2 + y^2 + 6x - 2y - 54 = 0$ but with radius half its size.

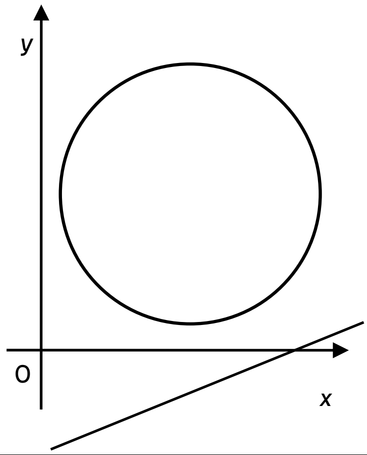
Intersection of Lines and Circles

As with parabolas, there are **three** possibilities when a line and a circle come into contact, and we can examine the roots of a rearranged quadratic equation to determine which has occurred. However:

We CANNOT make the circle and line equations equal to each other: the line equation must be substituted INTO the circle equation to obtain our quadratic equation!







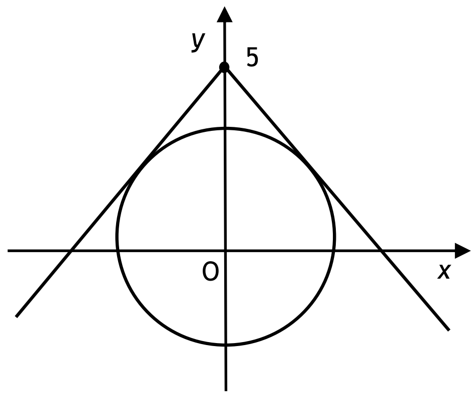
Two points of contact	One point of contact	No points of contact
2 distinct roots	**Equal roots**	**No real roots**
$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$

As with parabolas, the most common use of this technique is to show tangency.

Example 10: Find the coordinates of the points of intersection of the line $y = 2x - 1$ and the circle $x^2 + y^2 - 2x - 12y + 27 = 0$.

Example 11: Show that the line $y = 3x + 10$ is a tangent to the circle $x^2 + y^2 - 8x - 4y - 20 = 0$ and establish the coordinates of the point of contact.

Example 12: Find the equations of the tangents to the circle
$x^2 + y^2 = 9$ from the point (0, 5).

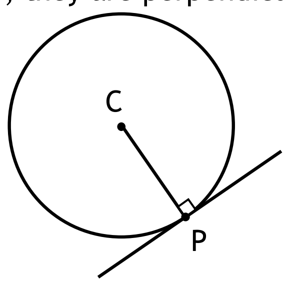


Tangents to Circles at Given Points

Remember: at the point of contact, the radius and tangent meet at 90° (i.e., they are perpendicular).

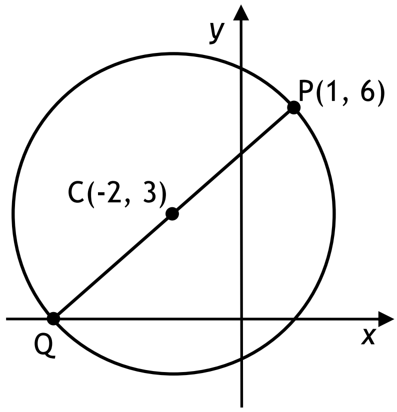
To find a tangent at a given point:

- **Find the centre of the circle**
- **Find the gradient of the radius (joining C and the given point)**
- **Find the gradient of the tangent (flip and make negative)**
- **Sub the gradient and the original point into $y - b = m(x - a)$**



Example 13: Find the equation of the tangent to $x^2 + y^2 - 14x + 6y - 87 = 0$ at the point (-2, 5).

Past Paper Example 1: A circle has centre C (-2, 3) and passes through point P (1, 6).



a) Find the equation of the circle.

b) PQ is a diameter of the circle. Find the equation of the tangent to this circle at Q.

Past Paper Example 2:

a) Show that the line with equation $y = 3 - x$ is a tangent to the circle with equation

$$x^2 + y^2 + 14x + 4y - 19 = 0$$

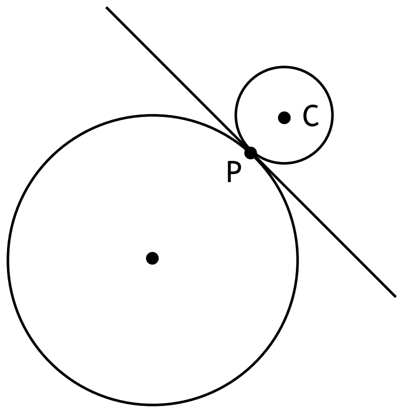
and state the coordinates of P, the point of contact.

b) Relative to a suitable set of coordinate axes, the diagram opposite shows the circle from a) and a second smaller circle with centre C.

The line $y = 3 - x$ is a common tangent at the point P.

The radius of the larger circle is three times the radius of the smaller circle.

Find the equation of the smaller circle.



Past Paper Example 3: Given that the equation

$$x^2 + y^2 - 2px - 4py + 3p + 2 = 0$$

represents a circle, determine the range of values of p.

Past Paper Example 4: Circle P has equation $x^2 + y^2 - 8x - 10y + 9 = 0$. Circle Q has centre (-2, -1) and radius $2\sqrt{2}$.

a) i) Show that the radius of circle P is $4\sqrt{2}$.
ii) Hence show that circles P and Q touch.

b) Find the equation of the tangent to circle Q at the point (-4, 1)