Roots of Complex Numbers Questions

1. (a) Find the modulus and argument of $1 + \mathrm{i}\sqrt{3}$

(b) Hence solve the equation $z^2 = 1 + \mathrm{i}\sqrt{3}$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$.

2. (a) Express the complex number $-1+\sqrt{3}\mathrm{i}$

in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r is real and $-\pi < \theta \le \pi$.

(b) (i) Verify that $2e^{\pi i/6}$ is a root of the equation $z^4 = 8(-1 + \sqrt{3}\,\mathrm{i})$

(ii) Find the other three roots of the above equation giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r is real and $-\pi < \theta \le \pi$.

3. In this question, give all answers in exact form with the arguments in radians.

(a) Write in modulus-argument form:

(i) $1+\sqrt{3}i$ (ii) $(1+\sqrt{3}i)^2$ (iii) $\dfrac{1}{1+\sqrt{3}i}$

(b) Solve the equation $z^4 = 16\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$, giving the solutions in modulus-argument form.

4. (a) (i) Verify that $z = 2\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$ is a root of the equation $z^4 = -16$.

(ii) Find the other three roots of this equation, giving each root in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r is real and $-\pi < \theta \le \pi$.

(iii) Illustrate the four roots of the equation by points on an Argand diagram.

(b) (i) Show that $\left(z - 2\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\right)\left(z - 2\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\right) = z^2 - 2\sqrt{2}\,z + 4$.

(ii) Express $z^4 + 16$ as the product of two quadratic factors with real coefficients.

5. A complex number, z, is said to be in polar form when it is written as $[r, \theta]$, where r is the modulus of z and θ is the argument of z with $-\pi < \theta \le \pi$.

(a) Express the complex number 512i in polar form.

(b) Hence determine, in polar form, the nine complex roots of the equation

$$z^9 - 512\mathrm{i} = 0$$

(c) On a sketch of the complex plane draw the nine points which represent these roots.

Explain why these nine points lie at the vertices of a regular nonagon. Find the area of this plane figure, giving your answer in the form $p \sin \alpha$, stating the values of p and α.

6. (a) The complex number –21.6 + 16.2i can be expressed in polar form $[r, \theta]$, where $r > 0$ and $-\pi < \theta \leq \pi$. Determine the exact value of r and give θ in radians to four decimal places.

(b) The roots of the equation $z^3 + 21.6 = 16.2\text{i}$ are α, β and γ.

(i) Find these roots, giving your answers in polar form.

(ii) The complex numbers α, β and γ are represented in the complex plane by the points A, B and C. Describe the geometrical relationship between A, B, and C and their position relative to the origin.

7. The complex number α is defined by

$$\alpha = \frac{2 - 10\text{i}}{3 - 2\text{i}}$$

(a) Show that $\alpha = 2 - 2\text{i}$

(b) Express α in the form $r\text{e}^{\text{i}\theta}$, where r is real and $-\pi < \theta \leq \pi$.

(c) Hence

(i) show that α^4 is real,

(ii) solve the equation

$$z^3 = \alpha$$

giving your answers in the form $r\text{e}^{\text{i}\theta}$, where r is real and $-\pi < \theta \leq \pi$.

Solutions to Roots of Complex Numbers Exam Questions

1. (a) modulus 2 ; argument $\frac{\pi}{3}$

   (b) $z = \sqrt{2}e^{\frac{i\pi}{6}}$ and $= \sqrt{2}e^{\frac{-i5\pi}{6}}$

2. (a) $r = 2,\ \theta = \frac{2}{3}\pi$

   (b) (ii) $z = 2e^{-5\pi i/6},\ 2e^{-\pi i/3},\ 2e^{2\pi i/3}$

3. (a) (i) $[2, \frac{\pi}{3}]$ (ii) $[4, \frac{2\pi}{3}]$ (iii) $\frac{[1,0]}{[2, \frac{\pi}{3}]} = \left[0.5, -\frac{\pi}{3}\right]$

   (b) $[2, \frac{\pi}{16}]\ [2, \frac{9\pi}{16}][2, \frac{17\pi}{16}][2, \frac{25\pi}{16}]$

4. (a)(ii) $z = 2e^{-\pi i/4},\ z = 2e^{\pm 3\pi i/4}$

   (b) (ii) $z^4 + 16 = (z^2 - 2\sqrt{2}\,z + 4)(z^2 + 2\sqrt{2} + 4)$

5. (a) $512i = \left[512, \frac{\pi}{2}\right]$

   (b) $r = 2$

   $\left[\text{Args. are} -\frac{5\pi}{6}, -\frac{11\pi}{18}, -\frac{7\pi}{18}, -\frac{\pi}{6}, -\frac{\pi}{18}, -\frac{5\pi}{18}, \frac{\pi}{2}, \frac{13\pi}{18}, -\frac{17\pi}{18}\right]$

   (c) $A = 18 \sin\frac{2\pi}{9}$

6. (a) $r = 27,\ \theta = 2.4981$

   (b) (i) z = (3, 0.8327), (3, 2.9271), (3, – 1.2617 or 5.0215)

   (ii) “All roots equidistant from 0” or “All lie on circle, centre 0” , etc

   Points form the vertices of an equilateral triangle

7. (a) Multiply numerator and denominator by the complex conjugate.

   (b) $r = 2\sqrt{2}$ or $\sqrt{8}$, $\theta = -\frac{\pi}{4}$

   (c) (ii) $z = \sqrt{2}\,e^{\frac{\pi i}{12}+2k\pi\frac{c}{3}}$ k = 0, ± 1