# Advanced Higher Mathematics Nightly Questions

You should attempt a short question every night during the school week. Your teacher will tell you what question to attempt each night.

The questions will usually take you between 5 and 10 minutes.

Doing a small amount of Advanced Higher Maths every night will improve your knowledge and confidence.

Your teacher will go over the answer to the nightly question in class the next day.

**1.** Express $\frac{4x-1}{(x-1)(x+2)}$ in partial fractions.

**2.** Given $f(x) = x(x+1)^4$, find $f'(x)$ in its fully factorised form.
[*hint*: use the product rule]

**3.** Given the curve $y = \dfrac{x}{x^2+1}$, find the gradient of the tangent to the curve where $x = 2$.
[*hint*: use the quotient rule]

**4.** Express $\dfrac{5x-3}{x^2+x-30}$ in partial fractions. [*hint*: factorise the denominator first]

**5.** Differentiate: (a) $f(x) = e^{x^2+1}$ (b) $g(x) = \ln\sqrt{x^2+1}$

[*hint*: you may wish to use the laws of logarithms to simplify $g(x)$ before differentiating]

**6.** Given $f(x) = \dfrac{x^2-1}{x^2+1}$, find $f'(x)$ and simplify your answer.

**7.** Express $\dfrac{x^2+6x-3}{x(x-1)^2}$ in partial fractions.

[*hint*: the denominator contains a repeated linear factor, so let $\dfrac{x^2+6x-3}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$]

**8.** Given that $y = e^{4x}\cos 2x$, find $\dfrac{dy}{dx}$ in its fully factorised form.

**9.** Differentiate: (a) $f(x) = \sin(e^{3x})$ (b) $g(x) = e^{\sin 2x}$

**10.** Given that the quadratic function $x^2+3$ is irreducible, express $\dfrac{x^2+2x+9}{(x-1)(x^2+3)}$ in partial fractions.

[*hint*: let $\dfrac{x^2+2x+9}{(x-1)(x^2+3)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+3}$]

**11.** Differentiate $f(x) = x^4\tan 2x$.

**12.** Differentiate: (a) $f(x) = \ln(x^4 + 1)$ (b) $g(x) = x\ln x, \quad x > 0$

**13.** Differentiate $f(x) = \dfrac{\sin 2x}{x^3}$, $x \neq 0$, and simplify your answer.

**14.** Express $\dfrac{x+3}{x(x-1)(x+1)}$ in partial fractions.

**15.** Differentiate: (a) $f(x) = \sin^{-1}(3x)$ (b) $g(x) = \tan^{-1}(x^2)$

**16.** (a) Given $f(x) = x^3 e^{x^2}$, find $f'(x)$ in its fully factorised form.

(b) Differentiate $f(x) = \dfrac{\tan^{-1} x}{1+x^2}$ and simplify your answer.

**17.** Differentiate $f(x) = \tan^{-1}(\sqrt{x-1})$, where $x > 1$, and simplify your answer.

**18.** $y$ is defined implicitly in terms of $x$ by the equation $y^2 + xy = 4$.
Use implicit differentiation to find $\dfrac{dy}{dx}$ in terms of $x$ and $y$.

**19.** Given $f(x) = \dfrac{\ln x}{x^2}$, $x > 0$, find $f'(x)$ and $f''(x)$ in their simplest form.

**20.** Given $y = 10^x$, use logarithmic differentiation to find $\dfrac{dy}{dx}$ in terms of $x$.

**21.** Given $f(x) = (x+1)(x-2)^3$, find $f'(x)$ in its fully factorised form.

**22.** Differentiate: (a) $f(x) = e^{\tan 4x}$ (b) $g(x) = (2x+1)e^{2x}$

**23.** (a) Given $y = \dfrac{x^4}{\sqrt{e^{2x}+1}}$, show that $\ln y = 4\ln x - \dfrac{1}{2}\ln(e^{2x}+1)$.

(b) Use logarithmic differentiation to find $\frac{dy}{dx}$ in terms of $x$.

**24.** A curve is defined by the parametric equations $x = 3\sin t$ and $y = 6\cos t$ for $0 \le t \le 2\pi$.

Find the gradient of the curve at the point where $t = \frac{\pi}{6}$.

[*hint*: use parametric differentiation]

**25.** (a) Use the discriminant to show that the quadratic function $x^2 + x + 2$ is irreducible.

(a) Hence express $\frac{6x-4}{(x-3)(x^2+x+2)}$ in partial fractions.

**26.** $y$ is defined implicitly in terms of $x$ by the equation $y^2 - x^2 = 2xy$.

Use implicit differentiation to find $\frac{dy}{dx}$ in terms of $x$ and $y$.

**27.** A curve is defined by the parametric equations $x = t^2 + 1$ and $y = 1 - 3t^3$ for all $t$.

(a) Use parametric differentiation to find $\frac{dy}{dx}$ in terms of $t$.

(b) Hence find the equation of the tangent to the curve at the point where $t = 2$.

**28.** Use the substitution $u = x^3 + 1$ to find $\int x^2(x^3+1)^5\,dx$.

**29.** Differentiate: (a) $f(x) = \ln(\cos 2x)$ (b) $g(x) = \frac{e^{3x}}{x^3}$

**30.** Use the substitution $u = x + 1$ to find $\int x\sqrt{x+1}\,dx$.

**31.** A curve has equation $y^3 + 3xy = 3x^2 - 5$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of $x$ and $y$.

(b) Find the equation of the tangent to the curve at the point A(2, 1).

**32.** Use the substitution $u = x^2 + 1$ to evaluate $\int_0^1 x(x^2+1)^9\,dx$.

**33.** A curve is defined by the parametric equations $x = t^2 + t - 1$ and $y = 2t^2 - t + 2$ for all *t*.
Find the equation of the tangent to the curve at the point where $t = -1$.

**34.** (a) Find $\int \frac{1}{\sqrt{25 - x^2}} dx$. (b) Evaluate $\int_0^2 \frac{1}{4 + x^2} dx$.

[*hint*: use $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$ and $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$]

**35.** (a) Express $\frac{5x - 4}{x^2 - 4x}$ in partial fractions. (b) Hence find $\int \frac{5x - 4}{x^2 - 4x} dx$.

**36.** Use integration by parts to find $\int xe^{4x} dx$.

**37.** Differentiate each function with respect to *x*:

(a) $f(x) = 3x^2 \cos x$ (b) $g(x) = \frac{3x + 1}{x - 1}, \quad x \neq 1$

**38.** Use integration by parts to evaluate $\int_0^{\pi} x \sin x dx$.

**39.** Find the general solution of the differential equation $e^y \frac{dy}{dx} = \sec^2 x$ by separating the variables.

**40.** In a town with a large population, a ‘flu virus is spreading rapidly.
The percentage, *P*, of the population infected *t* days after the initial outbreak satisfies the differential equation

$$\frac{dP}{dt} = kP, \qquad \text{where } k \text{ is a constant.}$$

(a) Find the general solution of this differential equation, expressing *P* in terms of *t*.

(b) Given that 0·5% of the population are initially infected and 2·5% are infected after 7 days, find the value of *k* correct to 4 decimal places.

**41.** A curve is defined by the parametric equations $x = t^2 + 1$ and $y = t(t - 3)^2$ for all *t*.

(a) Use parametric differentiation to find $\frac{dy}{dx}$ in terms of *t*.

(b) Hence find the coordinates of the two stationary points on the curve. [You do not need to determine the nature of the stationary points.]

**42.** (a) Express $\frac{11-2x}{x^2+x-2}$ in partial fractions.

(b) Hence evaluate $\int_3^5 \frac{11-2x}{x^2+x-2}dx$.

**43.** Use integration by parts twice to find $\int x^2 \sin x dx$.

**44.** Use an integrating factor to find the general solution of the first order linear differential equation

$$\frac{dy}{dx}+\frac{y}{x}=x, \quad x>0.$$

**45.** (a) Given that the quadratic function $x^2+5$ is irreducible, express $\frac{12x^2+20}{x(x^2+5)}$ in partial fractions. [*hint*: let $\frac{12x^2+20}{x(x^2+5)}=\frac{A}{x}+\frac{Bx+C}{x^2+5}$]

(b) Hence find $\int \frac{12x^2+20}{x(x^2+5)}dx$.

**46.** Use the substitution $u=x+4$ to evaluate $\int_0^5 x\sqrt{x+4}dx$.

**47.** Find the general solution of the differential equation $\frac{dy}{dx}=\frac{3y}{x-2}$ by separating the variables, expressing the solution in the form $y=f(x)$.

**48.** Differentiate and simplify where possible:

(a) $f(x) = e^{2x}\tan x$ (b) $g(x) = \dfrac{\cos 2x}{x^3}, \quad x \neq 0$

**49.** Find the general solution of the following second order differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x - 1.$$

**50.** A curve has equation $x^2 + 4xy + y^2 + 11 = 0$.

Find the value of $\dfrac{dy}{dx}$ at the point (2, –3). [*hint*: use implicit differentiation]

**51.** (a) Use the binomial theorem to expand $(x^2+2)^4$.

(b) Differentiate $f(x) = x^2\tan^{-1}(2x)$.

**52.** Expand and simplify $\left(a - \dfrac{2}{a}\right)^5$.

**53.** Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{2x}.$$

**54.** Solve the quadratic equation $z^2 + 2z + 17 = 0$ for the complex number *z*.
[*hint*: use the quadratic formula]

**55.** The complex numbers *z* and *w* are given by $z = 7 + 6i$ and $w = 2 + i$.

Find $zw$ and $\dfrac{z}{w}$ in the form $a + bi$.

**56.** A curve is defined by the parametric equations $x = \ln(1+t^2)$ and $y = \ln(1+2t^2)$ for all *t*.

Use parametric differentiation to find $\dfrac{dy}{dx}$ in terms of *t*.

**57.** Given $y = x^x$, where $x > 0$, use logarithmic differentiation to find $\dfrac{dy}{dx}$ in terms of *x*.

**58.** (a) Given the complex number $z = 2 + 3i$, express $\bar{z}$ and $z^2$ in the form $a + bi$, where $\bar{z}$ is the complex conjugate of *z*.

(b) The complex number *w* is given by $w = \dfrac{(1+2i)^2}{7-i}$.

Express *w* in the form $a + bi$.

**59.** Find the particular solution of the differential equation $\dfrac{dy}{dx} = 2xy$, given that $y = 3$ when $x = 0$. [*hint*: separate the variables]

**60.** An arithmetic sequence begins 2, 6, 10, 14, ...

(a) Find $u_{30}$, the 30th term of this sequence.

(b) Find $S_{30}$, the sum of the first 30 terms of this sequence.

**61.** A mathematical biologist believes that the differential equation $x\dfrac{dy}{dx} - 3y = x^4$ models a process. Use an integrating factor to find the general solution of the differential equation.

[*hint*: first rewrite the differential equation in the standard form $\dfrac{dy}{dx} + P(x)y = Q(x)$.]

**62.** (a) Express $\dfrac{4x-1}{2x^2 - x - 6}$ in partial fractions.

(b) Hence find $\displaystyle\int \frac{4x-1}{2x^2 - x - 6}dx$.

**63.** (a) Write down the common ratio of the geometric sequence beginning 24, –12, 6, –3, ...

(b) Explain why the associated geometric series has a sum to infinity.

(c) Find the sum to infinity.

**64.** (a) Verify that $z = 1 + i$ is a root of the equation $z^3 + 16z^2 - 34z + 36 = 0$.

(b) Write down a second root of the equation and then find the third root.

**65.** (a) Use the substitution $u = 2x$ (or otherwise) to find $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}dx$.

(b) Evaluate $\int_0^{\frac{\pi}{12}} \sec^2 4x\,dx$.

**66.** Relative to suitable coordinate axes, the position of a golf ball at time *t* seconds is (*x*, *y*), where *x* and *y* are functions of *t* given by

$$x = 10t, \qquad y = 20t - 4t^2 \qquad (0 \le t \le 5),$$

and *x* and *y* are measured in metres.

(a) Use parametric differentiation to obtain an expression for $\frac{dy}{dx}$ in terms of *t*.

(b) The speed, *v* metres per second, of the golf ball at time *t* seconds is given by

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Use this formula to find the speed of the golf ball after 2 seconds.

**67.** The first term of an arithmetic sequence is 2 and the 20$^{\text{th}}$ term is 97.

(a) Find the common difference.
(b) Find the sum of the first 50 terms of this sequence.

**68.** (a) Use the standard formula for $\sum_{r=1}^{n} r$ to find an expression for $\sum_{r=1}^{n}(3r-2)$ in terms of *n*.

(b) Hence, or otherwise, evaluate $\sum_{r=1}^{16}(3r-2)$.

**69.** (a) Write down and simplify the general term in the binomial expansion of $(x^2+3x)^8$.

(b) Hence, or otherwise, obtain the coefficient of $x^{13}$ in the expansion.

[*hint*: the general term in the binomial expansion of $(a+b)^n$ is $\binom{n}{r}a^{n-r}b^r$ ]

**70.** The equation of a curve is $y = \dfrac{x^2+4x+5}{x+1}$, $x \ne -1$.

(a) Write down the equation of the vertical asymptote.
(b) Find the equation of the non-vertical asymptote. [*hint*: use long division first]

**71.** Obtain the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2.$$

**72.** The equation $x^2y + xy^3 = 5$ defines *y* implicitly as a function of *x*.
Use implicit differentiation to find $\frac{dy}{dx}$ in terms of *x* and *y*.

**73.** Use integration by parts to find: (a) $\int xe^{3x}dx$ (b) $\int x^2 \ln x dx$

**74.** (a) Find the Maclaurin series for $f(x) = e^{2x}$ up to and including the term in $x^4$.
(b) Find the Maclaurin series for $g(x) = \cos 3x$ up to and including the term in $x^4$.
(c) **Hence** find the Maclaurin series for $h(x) = e^{2x}\cos 3x$ up to and including the term in $x^3$.

**75.** Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x}.$$

**76.** (a) Expand and simplify $\left(x^3 - \frac{2}{x^2}\right)^4$.

(b) Differentiate: (i) $f(x) = \ln(1+e^{2x})$ (ii) $g(x) = \frac{\sin x}{x}$, $x \neq 0$

**77.** Differentiate $f(x) = e^{\tan x}\cos^2 x$ and simplify your answer.

**78.** (a) Given the complex number $z = 1 - \sqrt{3}i$, write down $\bar{z}$ and express $(\bar{z})^2$ in the form $a + bi$, where $\bar{z}$ is the complex conjugate of *z*.

(b) The function $f$ is defined on the set of all real numbers by $f(x) = x^3 \sin x$.
By considering $f(-x)$, determine whether the function $f$ is odd, even or neither.

**79.** Find $\int \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx$. [*hint*: use partial fractions]

**80.** (a) Given $y = \frac{x^3 e^{\sin x}}{\sqrt{x^2+1}}$, show that $\ln y = 3\ln x + \sin x - \frac{1}{2}\ln(x^2+1)$.

(b) Use logarithmic differentiation to find $\frac{dy}{dx}$ in terms of $x$.

**81.** Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 8\sin x + 19\cos x.$$

**82.** (a) Given $xy - x = 4$, use implicit differentiation to find $\frac{dy}{dx}$ in terms of $x$ and $y$.

(b) Hence find $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$.

**83.** A curve is defined by the parametric equations $x = \frac{1}{2}t^2 + 2t$ and $y = \frac{1}{3}t^3 - 3t$ for all $t$.

(a) Use parametric differentiation to find $\frac{dy}{dx}$ in terms of $t$.

(b) Hence find $\frac{d^2y}{dx^2}$ in terms of $t$.

**84.** Express $\frac{x^2+6}{x^3+2x}$ in partial fractions. [*Hint*: start by factorising the denominator]

**85.** Use integration by parts twice to find $\int x^2 e^{2x}\,dx$.

**86.** (a) Write down and simplify the general term in the binomial expansion of

$$\left(x^2+\frac{2}{x}\right)^{10}.$$

(b) Hence, or otherwise, obtain the coefficient of $x^8$ in the expansion.

[*hint*: the general term in the binomial expansion of $(a+b)^n$ is $\binom{n}{r}a^{n-r}b^r$ ]

**87.** A solid is formed by rotating the area under the curve $y=\sqrt{4-x^2}$ between $x=0$ and $x=2$ through $360°$ about the *x*-axis.
Calculate the volume of the solid of revolution that is formed.

**88.** Use Gaussian elimination to solve the following system of equations.

$$\begin{aligned} x + \ \ y + 3z &= 2 \\ 2x + \ \ y + \ \ z &= 2 \\ 3x + 2y + 5z &= 5 \end{aligned}$$

**89.** The equation of a curve is $y=\dfrac{x^2+3x+6}{x+2}$, $x\neq -2$.

(a) Write down the equation of the vertical asymptote.
(b) Find the equation of the non-vertical asymptote. [*hint*: use long division first]
(c) Find the coordinates of the stationary points on the curve and justify their nature.

**90.** The velocity, *v* metres per second, of a particle moving in a straight line, *t* seconds after passing through a fixed point O is

$$v=6t^2-5t+3, \qquad t\geq 0.$$

(a) Find the acceleration of the particle when $t = 3$.
(b) Find the displacement of the particle from O when $t = 4$.

**91.** (a) Given the complex number $z=-1+\sqrt{3}i$, find the modulus and argument of *z* and hence write *z* in polar form.

(b) The function *f* is defined on the set of all real numbers by $f(x)=x^2\cos x$.
By considering $f(-x)$, determine whether the function *f* is odd, even or neither.

**92.** (a) Express $\dfrac{1}{y(1-y)}$ in partial fractions.

(b) Show that the general solution of the differential equation

$$\frac{dy}{dx} = y(1-y)$$

can be expressed in the form

$$\ln y - \ln(1-y) = x + C$$

for some constant $C$.

(c) Given that $y = \frac{1}{2}$ when $x = 0$, find the value of $C$ and hence show that $y = \frac{e^x}{1+e^x}$.

**93.** (a) Expand and simplify $\left(3x - \frac{2}{x}\right)^4$.

(b) Use Gaussian elimination to solve the following system of equations.

$$x + y = 4$$
$$x + 2y - z = 5$$
$$3y - 2z = 6$$

**94.** Let $A$ be the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

(a) Show that $A^2 = pA$ for some constant $p$.

(b) Hence, or otherwise, determine the value of $q$ such that $A^4 = qA$.

**95.** (a) Write the complex number $z = \sqrt{3} + i$ in polar form.

(b) Use de Moivre's theorem to find $z^6$ in the form $a + bi$.

**96.** A car begins from rest and travels along a straight road.

The velocity, $v$ metres per second, of the car is given by $v = \frac{100t}{2t+11}$, where $t$ is the time in seconds.

Find the acceleration of the car after 4 seconds.

**97.** Consider the system of equations

$$x + y + z = 2$$

$$4x + 3y - \lambda z = 4$$
$$5x + 6y + 8z = 11.$$

(a) Use Gaussian elimination to express $z$ in terms of $\lambda$ and hence state the value of $\lambda$ for which the system has no solution.

(b) Solve the system of equations when $\lambda = 2$.

**98.** Matrices $A$ and $B$ are defined by $A = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} q & -6 \\ 1 & 3 \end{pmatrix}$.

(a) Write down the matrix $A'$, where $A'$ denotes the transpose of $A$.
(b) Find the values of $p$ and $q$ if $B = 3A'$.

**99.** (a) Write the complex number $2 + 2\sqrt{3}i$ in polar form.

(b) Hence find the two complex numbers $z$ for which $z^2 = 2 + 2\sqrt{3}i$, expressing each root in the form $r(\cos\theta + i\sin\theta)$.

**100.** Let $I = \int e^{2x} \cos x dx$.

By integrating by parts twice, show that $I = \frac{1}{5}e^{2x}(\sin x + 2\cos x) + C$.

**101.** The area bounded by the curve with equation $y = e^{-2x}$ between $x = 0$ and $x = 1$ is rotated through $360°$ about the $x$-axis.

Find the exact value of the volume of the solid formed.

**102.** A particle moves along a straight line.
The velocity, $v$ metres per second, of the particle after $t$ seconds is given by

$$v = t^3 - 12t^2 + 32t.$$

(a) Find an expression for the acceleration of the particle after $t$ seconds.

(b) At time $t = 0$, the particle is at the origin $O$.

(i) Find an expression for the displacement of the particle from $O$ after $t$ seconds.
(ii) The particle returns to $O$ after $T$ seconds. Find $T$.

**103.** Let $A$ be the $2 \times 2$ matrix $A = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix}$.

(a) Find the values of $\lambda$ such that the matrix is singular.
[*Hint*: remember that $A$ is singular when $\det A = 0$]

(b) Write down the inverse matrix matrix $A^{-1}$ when $\lambda = 3$.

**104.** (a) Solve the following system of equations using Gaussian elimination.

$$\begin{aligned} 2x + y + 3z &= -5 \\ -x \quad\; + 2z &= -3 \\ 4x + 3y + z &= 3 \end{aligned}$$

(b) Use the substitution $u = x^4$ to find $\int \frac{x^3}{1+x^8}\,dx$.

**105.** (a) Given $\boldsymbol{u} = -2\boldsymbol{i} + 5\boldsymbol{k}$, $\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{w} = -\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$, calculate $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$.

(b) Find the point of intersection of the line

$$\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2}$$

and the plane with equation $2x + y - z = 4$.

[*Hint*: express the equation of the line in parametric form first]

**106.** The complex number $z$ is such that $|z-2| = |z+i|$, where $z = x + yi$.
Find the equation of the locus of z in the form $ax + by + c = 0$.

**107.** When a valve is opened, the rate at which water drains from a pool is proportional to the square root of the depth of the water.

This can be represented by the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{10}$, where $h$ is the depth (in metres) of the water and $t$ is the time (in minutes) elapsed since the valve was opened.

(a) Show that $\sqrt{h} = C - \frac{t}{20}$ for some constant $C$. [*hint*: separate the variables]

(b) Given that the water was initially 9 metres deep when the valve was opened, find the value of $C$ and hence find the time taken to drain the pool.

**108.** Obtain the equation of the plane passing through the points $A(-2, 1, -1)$, $B(1, 2, 3)$ and $C(3, 0, 1)$.

[*Hint*: find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then the normal vector is $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ ]

**109.** Find the values of *m* for which the matrix

$$A = \begin{pmatrix} m & 1 & 1 \\ 0 & m & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

is singular.

[*Hint*: remember that *A* is singular when $\det A = 0$ ]

**110.** (a) Write down and simplify the general term in the binomial expansion of $\left(2x + \frac{1}{x^2}\right)^9$.

(b) Hence, or otherwise, obtain the term independent of *x* in the expansion.

[*hint*: the general term in the binomial expansion of $(a+b)^n$ is $\begin{pmatrix} n \\ r \end{pmatrix} a^{n-r} b^r$ ]

**111.** Consider the system of equations

$$\begin{aligned} 4x + 6z &= 1 \\ 2x - 2y + 4z &= -1 \\ -x + y + \lambda z &= 2 \end{aligned}$$

Use Gaussian elimination to express $z$ in terms of $\lambda$ and hence state the value of $\lambda$ for which the system has no solution.

**112.** The lines $L_1$ and $L_2$ are given by the equations

$$\frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1} \quad \text{and} \quad \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2},$$

respectively.

(a) Show that $L_1$ and $L_2$ intersect and find the point of intersection.
[*Hint*: first express the equations of both lines in parametric form]

(b) Calculate the acute angle between the lines.

**113.** (a) Use the Euclidean algorithm to obtain the greatest common divisor of 1139 and 629.

(b) Express the greatest common divisor in the form $1139p + 629q$ for integers $p$ and $q$.

**114.** (a) Use a direct proof to prove that the product of an odd integer and an even integer is always even.

[*Hint*: Let $a$ be an odd integer, so that $a = 2k + 1$ for some integer $k$.
Let $b$ be an even integer, so that $b = 2m$ for some integer $m$.]

(b) Prove by contradiction that if $n^2$ is even, then $n$ is also even, where $n$ is a natural number.

[*Hint*: Let $n^2$ be even and assume that $n$ is odd.]

**115.** Prove by induction that for all positive integers $n$, $\sum_{r=1}^{n} 4r^3 = n^2(n+1)^2$.

**116.** (a) Verify that $z = 1 + 2i$ is a root of the equation $z^3 + 3z^2 - 5z + 25 = 0$.
(b) Hence find all the roots of this equation.

**117.** The twenty-first term of an arithmetic sequence is 37.
The sum of the first twenty terms of the sequence is 320.
What is the sum of the first ten terms?

**118.** Three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are given by

$$\boldsymbol{u} = 5\boldsymbol{i} + 13\boldsymbol{j}, \quad \boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}, \quad \boldsymbol{w} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}.$$

Calculate $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ and interpret your answer geometrically.

**119.** The $3 \times 3$ matrix $A$ is given by $A = \begin{pmatrix} \lambda & 1 & -3 \\ 0 & 1 & 2 \\ 1 & -2 & 1 \end{pmatrix}$.

(a) Obtain an expression for $\det A$ in terms of $\lambda$.

(b) Hence find the value of $\lambda$ for which the matrix $A$ is singular.

**120.** Prove by induction that for all positive integers $n$, $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$.

**121.** An arithmetic sequence begins 5, 7, 9, 11, ...

(a) Find a formula for $S_n$, the sum of the first $n$ terms of this sequence.

(b) Hence find the value of $n$ for which $S_n = 320$.

**122.** Plane $\pi_1$ has equation $2x + 3y + z = 5$ and plane $\pi_2$ has equation $x + y - z = 0$. Calculate the size of the acute angle between the planes $\pi_1$ and $\pi_2$.

[*Hint*: the angle between two planes is the angle between their normal vectors]

**123.** Prove by induction that for all positive integers $n$, $5^n + 3$ is divisible by 4.

**124.** Show that the line of intersection of the planes $x + y - z = 6$ and $2x - 3y + 2z = 2$ has parametric equations

$$x = \lambda, \quad y = 4\lambda - 14, \quad z = 5\lambda - 20.$$

[*Hint*: let $x = \lambda$ in the equation of each plane and solve the system of equations for $y$ and $z$]

**125.** Let $A$ be the $2 \times 2$ matrix $A = \begin{pmatrix} 4 & \lambda \\ -2 & 1 \end{pmatrix}$.

(a) Find $A^2$.

(b) Find the value of $\lambda$ for which $A^2$ is singular.
[*Hint*: remember that $A^2$ is singular when $\det(A^2) = 0$]

**126.** (a) Write down the $2 \times 2$ matrix $M_1$ associated with an anti-clockwise rotation of $\frac{\pi}{2}$ radians about the origin.

(b) Write down the matrix $M_2$ associated with reflection in the $y$-axis.

(c) Find the $2 \times 2$ matrix, $M_3$, associated with an anti-clockwise rotation of $\frac{\pi}{2}$ radians about the origin followed by reflection in the $y$-axis.

(d) What single transformation is associated with the matrix $M_3$?

**127.** (a) Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12x^2 + 2x - 5.$$

(b) When $x = 0$, $y = -6$ and $\frac{dy}{dx} = 3$.

Find the particular solution of the differential equation.

**128.** (a) Obtain the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin x .$$

(b) Hence find the particular solution for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.