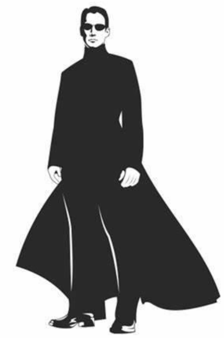


AH Maths – Matrices #2

1. Explain what it means for a system of equations to be

   (a) Inconsistent (b) Redundant

2. Solve each system of equations using an augmented matrix. For the redundant system give the answer in general form. For the inconsistent system explain why there is no solution.

   (a) $x + y + z = 10$
   $3x - y - z = 0$
   $2x + y - z = 9$

   (b) $x + 2y + z = 10$
   $3x - y - z = 1$
   $7x - z = 8$

   (c) $x + 2y + z = 10$
   $3x - y - z = 1$
   $7x - z = 12$

3. Explain what it means for a matrix to be

   (a) Square (b) Singular (c) Orthogonal

4. Find x and y so the matrices $A = \begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & y \end{pmatrix}$ are orthogonal

5. For matrices $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 4 \\ 0 & 1 \end{pmatrix}$

   (a) Verify that $(AB)^T = B^T A^T$
   (b) Verify that $\det(AB) = \det(A)\det(B)$
   (c) Verify that $(AB)^{-1} = B^{-1}A^{-1}$

6. Explain what it means for a system of equations to be ill-conditioned

7. A system of equations is formed by choosing two out of three equations below.
   Which choice of two equations results in a system that is ill-conditioned?

$$x + 3y = 10, \quad 4x + 5y = 2, \quad 8x + 9y = 11$$

8. Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -5 \\ 3 & 2 & -2 \end{pmatrix}$

9. (a) Write down the matrix that represents a rotation 60 degrees anti-clockwise about the origin
   (b) Write down the matrix that represents a reflection in the y-axis
   (c) Find the matrix that represents the 60 degree rotation, followed by the reflection in the y -axis.

AH Maths – Matrices #2 – Solutions

1. Explain what it means for a system of equations to be

   (a) Inconsistent – there are no values that satisfy all the equations

   (b) Redundant – there are more equations than necessary to fully solve the system

2. Solve each system of equations using an augmented matrix. For the redundant system give the answer in general form. For the inconsistent system explain why there is no solution.

   (a) $x + y + z = 10$
   $3x - y - z = 0$
   $2x + y - z = 9$

   $$x = \frac{5}{2}, y = \frac{23}{4}, z = \frac{7}{4}$$

   (b) $x + 2y + z = 10$ (1)
   $3x - y - z = 1$ (2)
   $7x - z = 8$ (3)
   No solutions. System is inconsistent. For example, doing (1)+2(2)-(3) leads to 0 = 4

   (c) $x + 2y + z = 10$
   $3x - y - z = 1$
   $7x - z = 12$
   Letting $x = t$ the general solution is $x = t, y = 11 - 4t, z = 7t - 12$

3. Explain what it means for a matrix to be

   (a) Square – same number of rows and columns
   (b) Singular – determinant is zero, or equivalently no inverse exists
   (c) Orthogonal – $A^T A = I$ or equivalently $A^T = A^{-1}$

4. Find x and y so the matrices $A = \begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & y \end{pmatrix}$ are orthogonal

   $$A^T A = I$$
   $$\begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
   $$\begin{pmatrix} 1 + x^2 & -x \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
   $$x = 0$$

   $$B^T B = I$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & y \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{\sqrt{3}}{4} + \frac{1}{2}y \\ \frac{\sqrt{3}}{4} + \frac{1}{2}y & \frac{1}{4} + y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$y = -\frac{\sqrt{3}}{2}$$

5. For matrices $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 4 \\ 0 & 1 \end{pmatrix}$

(a) Verify that $(AB)^T = B^T A^T$ they are both $\begin{pmatrix} -2 & -4 \\ 7 & 12 \end{pmatrix}$

(b) Verify that $\det(AB) = \det(A)\det(B)$ they are both 4

(b) Verify that $(AB)^{-1} = B^{-1}A^{-1}$ they are both $\frac{1}{4}\begin{pmatrix} 12 & -7 \\ 4 & -2 \end{pmatrix}$

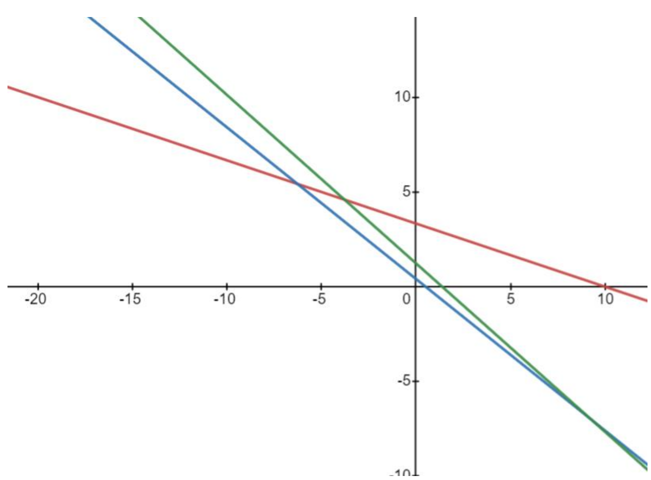
6. Explain what it means for a system of equations to be ill-conditioned

The solution is sensitive to a small relative change in the initial conditions, which produce a large relative change in the solution. This means any uncertainty in coefficients or right-hand-sides of equations makes the solution unpredictable

7. A system of equations is formed by choosing two out of three equations below. Which choice of two equations results in a system that is ill-conditioned?

$$x + 3y = 10, \quad 4x + 5y = 2, \quad 8x + 9y = 11$$

The last two equations. These are nearly parallel (as seen by the gradient), hence the meeting point of the two lines will vary greatly if there is any change in the coefficients.



8. Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -5 \\ 3 & 2 & -2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -5 \\ 3 & 2 & -2 \end{pmatrix}^{-1} = \frac{1}{10}\begin{pmatrix} -10 & -10 & 10 \\ 13 & 11 & -8 \\ -2 & -4 & 2 \end{pmatrix}$$

9. (a) Write down the matrix that represents a rotation 60 degrees anti-clockwise about the origin

$$\begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(b) Write down the matrix that represents a reflection in the y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(c) Find the matrix that represents the 60 degree rotation, followed by the reflection in the y-axis.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Note that when applying two transformations the one that happens first goes on the *right*