AH Maths – Complex Numbers #2

1. Solve: $z\bar{z} = z + 2$

2. Use DeMoivre's theorem to find

   (a) z^3, where $z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
   (b) z^3, where $z = 2 + 2i$
   (c) $z^{\frac{1}{2}}$, where $z = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

3. Use DeMoivre's theorem to find all solutions

   (a) $z^3 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
   (b) $z^3 = 1$
   (c) $z^4 = -64$

4. In each case find the Loci of complex numbers that satisfy the equation

   (a) $|z| = 1$
   (b) $(z - 3)^2 = -16$
   (c) $|z| = |z - 3|$
   (d) $|z - 4| < 2$

5. Solve $z^2 = 21 - 20i$

AH Maths – Complex Numbers #2 – Solutions

1. Solve $z\,\bar{z} = z + 2$

   Replace $z = x + iy$

   $$(x + iy)(x - iy) = x + iy + 1$$
   $$x^2 + y^2 = x + 2 + iy$$

   Equating imaginary parts

   $$0 = iy$$
   $$0 = y$$

   Equating real parts

   $$x^2 - y^2 = x + 2$$
   $$x^2 = x + 2$$
   $$x = -1, 2$$

   Hence $z = -1$ or $z = 2$

2. Use DeMoivre's theorem to find

   (a) z^3, where $z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

   $$z^3 = 3^3\left(\cos 3 \times \frac{\pi}{6} + i\sin 3 \times \frac{\pi}{6}\right)$$
   $$z^3 = 27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$
   $$z^3 = 27i$$

   (b) z^3, where $z = 2 + 2i$

   Put into polar form first

   $$z = 2 + 2i$$
   $$z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

   Now apply De Moivre's Theorem

   $$z^3 = \left(2\sqrt{2}\right)^3\left(\cos 3 \times \frac{\pi}{4} + i\sin 3 \times \frac{\pi}{4}\right)$$
   $$z^3 = 16\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$
   $$z^3 = 16\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
   $$z^3 = -16 + 16i$$

   (c) $z^{\frac{1}{2}}$, where $z = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

   First solution by taking direct square root

$$z^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(\cos\frac{1}{2}\times\frac{\pi}{2} + i\sin\frac{1}{2}\times\frac{\pi}{2}\right)$$
$$z^{\frac{1}{2}} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$z^{\frac{1}{2}} = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
$$z^{\frac{1}{2}} = \sqrt{2} + i\sqrt{2}$$

To find second solution, add 2π radians to the original equation

$$z = 4\left(\cos\frac{\pi}{2} + 2\pi + i\sin\frac{\pi}{2} + 2\pi\right)$$
$$z = 4\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$$

Now find square root

$$z^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(\cos\frac{5\pi}{2}\times\frac{1}{2} + i\sin\frac{5\pi}{2}\times\frac{1}{2}\right)$$
$$z^{\frac{1}{2}} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$
$$z^{\frac{1}{2}} = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$
$$z^{\frac{1}{2}} = -\sqrt{2} - i\sqrt{2}$$

3. Use DeMoivre's theorem to find all solutions

(a) $z^3 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

First solution just finding cube root

$$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$z_1 = \sqrt{3} + i$$

Second solution start by adding 2π to original angle

$$z^3 = 8\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$$
$$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$
$$z_2 = -\sqrt{3} + i$$

Third solution start by adding another 2π to original angle

$$z^3 = 8\left(\cos\frac{9\pi}{2} + i\sin\frac{9\pi}{2}\right)$$
$$z = 2\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right)$$
$$z_3 = -2$$

(b) $z^3 = 1$

Write in polar form

$$z^3 = (\cos 0 + i \sin 0)$$

First solution just finding cube root

$$z = \left(\cos\frac{0}{3} + i \sin\frac{0}{3}\right)$$

$$z_1 = 1$$

Second solution start by adding 2π to original angle

$$z^3 = (\cos 2\pi + i \sin 2\pi)$$

$$z = \left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right)$$

$$z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Third solution start by adding another 2π to original angle

$$z^3 = (\cos 4\pi + i \sin 4\pi)$$

$$z = \left(\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}\right)$$

$$z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

(c) $z^4 = -64$

Write in polar form

$$z^4 = 64(\cos \pi + i \sin \pi)$$

First solution just finding cube root

$$z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$$

$$z_1 = 2 + 2i$$

Second solution start by adding 2π to original angle

$$z^4 = 64(\cos 3\pi + i \sin 3\pi)$$

$$z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$$

$$z_2 = -2 + 2i$$

Third solution start by adding another 2π to original angle

$$z^4 = 64(\cos 5\pi + i \sin 5\pi)$$

$$z = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right)$$

$$z_3 = -2 - 2i$$

Fourth solution start by adding another 2π to original angle

$$z^4 = 64(\cos 7\pi + i \sin 7\pi)$$

$$z = 2\sqrt{2}\left(\cos\frac{7\pi}{4} + i \sin\frac{7\pi}{4}\right)$$

$$z_4 = 2 - 2i$$

4. In each case find the Loci of complex numbers that satisfy the equation

(a) $|z| = 1$

Geometric method:

$$|z| = 1$$

$$|z - 0| = 1$$

This is all the points with distance from the origin of 1, hence a circle radius 1 centre the origin.

Algebraic method:

$$|z| = 1$$
$$|x + iy| = 1$$
$$\sqrt{x^2 + y^2} = 1$$
$$x^2 + y^2 = 1$$

This is a circle of radius 1, centre the origin.

(b) $(z - 3)^2 = -16$

Algebraic method:

$$z - 3 = \pm 4i$$
$$x + iy - 3 = \pm 4i$$

Real part:

$$x - 3 = 0$$
$$x = 3$$

Imaginary part:

$$y = \pm 4$$

Solutions

$$z = 3 + 4i, \qquad z = 3 - 4i$$

(c) $|z| = |z - 3|$

Geometric method:

$$|z - 0| = |z - 3|$$

This is all the points where the distance fom (0,0) is the same as the distance from (3,0). This is the perpendicular bisector of (0,0) and (3,0), hence the vertical line $x = 1.5$

Algebraic method:

$$|x + iy| = |x + iy - 3|$$
$$\sqrt{x^2 + y^2} = \sqrt{(x - 3)^2 + y^2}$$
$$x^2 + y^2 = (x - 3)^2 + y^2$$
$$x^2 = x^2 - 6x + 9$$
$$6x = 9$$

$$x = 1.5$$

This is a vertical line $x = 1.5$

(d) $|z - 4| < 2$

Geometric method: This is all the points at distance less than 2 from (4,0), so the interior of a circle of radius 2 centred at (4,0)

Algebraic method:

$$|x + iy - 4| < 2$$
$$\sqrt{(x-4)^2 + y^2} < 2$$
$$(x-4)^2 + y^2 < 4$$

This is is the interior of a circle of radius 2, centred at (4,0).

5. Solve $z^2 = 21 - 20i$

$$(a + bi)^2 = 21 - 20i$$
$$a^2 - b^2 + 2abi = 21 - 20i$$

Real part:

$$a^2 - b^2 = 21$$

Imaginary part:

$$2ab = -20$$
$$ab = -10$$
$$b = -\frac{10}{a}$$

Substitute into real part:

$$a^2 - \left(\frac{10}{a}\right)^2 = 21$$
$$a^2 - \frac{100}{a^2} = 21$$

Let $c = a^2$

$$c - \frac{100}{c} = 21$$
$$c^2 - 100 = 21c$$
$$c^2 - 21c - 100 = 0$$
$$(c - 25)(c + 4) = 0$$

Since $c = a^2$ it must be positive, so

$$c = 25$$
$$a = \pm 5$$

Find

Find b

$$a = 5, b = -2$$
$$a = -5, b = 2$$

State solutions

$$z = 5 - 2i, \qquad z = -5 + 2i$$