Advanced Higher Maths
Binomial Theorem

1. Evaluate the following using factorials

   (a) $\binom{7}{4}$
   (b) $\binom{20}{11}$
   (c) The number of ways of choosing three people out of ten

2. Evaluate the following (non-calculator)

   (a) $\binom{100}{1}$
   (b) $\binom{100}{99}$
   (c) $\binom{100}{0}$

3. Simplify each expression (non-calculator)

   (a) $\frac{5!}{4!}$
   (b) $\frac{15!}{14!}$
   (c) $\frac{21!}{22!}$

4. Expand fully

   (a) $(x+y)^3$
   (b) $(2+3x)^3$
   (c) $\left(\frac{2}{x}-x^2\right)^3$

5. In each case, find the coefficient of x^5

   (a) $(x+2)^7$
   (b) $(2x-3)^7$

6. In each case, find the coefficient of x^2

   (a) $\left(3x^2+\frac{2}{x}\right)^7$
   (b) $\left(2x-\frac{2}{x^3}\right)^{10}$

Advanced Higher Maths
Binomial Theorem – Solutions

1. Evaluate the following using factorials

(a) $\binom{7}{4} = \frac{7!}{4!3!} = 35$

(b) $\binom{20}{11} = \frac{20!}{11!9!} = 167\,960$

(c) The number of ways of choosing three people out of ten = $\binom{10}{3} = \frac{10!}{7!3!} = 120$

2. Evaluate the following (non-calculator)

(a) $\binom{100}{1} = 100$ (as there is 1 way to choose one thing out of one hundred)

(b) $\binom{100}{99} = 100$ (as picking 99 things out of 100 is the same as leaving out 1 thing out of 100, so this is the same as the question in (a))

(c) $\binom{100}{0} = 1$ (as there is one way to pick *no* things out of one hundred)

3. Simplify each expression (non-calculator)

(a) $\frac{5!}{4!} = \frac{5\times4\times3\times2\times1}{4\times3\times2\times1} = \frac{5\times\cancel{4\times3\times2\times1}}{\cancel{4\times3\times2\times1}} = 5$

(b) $\frac{15!}{14!} = \frac{15\times14\times13\times\ldots\times2\times1}{14\times13\times\ldots\times2\times1} = 15$

(c) $\frac{21!}{22!} = \frac{21\times20\times19\times\ldots\times2\times1}{22\times21\times21\ldots\times2\times1} = \frac{1}{22}$

4. Expand fully

(a) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

(b) $(2+3x)^3 = 2^3 + 3(2^2)(3x) + 3(2)(3x)^2 + (3x)^3$

$= 8 + 36x + 54x^2 + 27x^3$

(c) $\left(\frac{2}{x} - x^2\right)^3 = \left(\frac{2}{x}\right)^3 + 3\left(\frac{2}{x}\right)^2(-x^2) + 3\left(\frac{2}{x}\right)(-x^2)^2 + (-x^2)^3$

$= \frac{8}{x^3} - 12 + 6x^3 - x^6$

5. In each case, find the coefficient of x^5

(a) $(x+2)^7$

General term is $x^r 2^{7-r}\binom{7}{r}$

Term in x^5 has $r = 5$ so is $x^5 2^2 \binom{7}{2} = 84x^5$ so coefficient is 84

(b) $(2x-3)^7$

General term is $(2x)^r(-3)^{7-r}\binom{7}{r}$

Term in x^5 has $r=5$ so is $(2x)^5(-3)^2\binom{7}{2}=6048x^5$ so coefficient is 6048

6. In each case, find the coefficient of x^2

(a) $\left(3x^2+\frac{2}{x}\right)^7$

General term is $(3x^2)^r\left(\frac{2}{x}\right)^{7-r}\binom{7}{r}$

Just looking at powers of x this is

$$(x^2)^r\left(\frac{1}{x}\right)^{7-r}$$
$$=x^{2r}(x^{-1})^{7-r}$$
$$=x^{2r}x^{r-7}$$
$$=x^{3r-7}$$

In order to get the coefficient of x^2 we solve

$$3r-7=2$$
$$3r=9$$
$$r=3$$

So the whole thing

$$(3x^2)^r\left(\frac{2}{x}\right)^{7-r}\binom{7}{r}$$

With $r=3$

$$(3x^2)^3\left(\frac{2}{x}\right)^{7-3}\binom{7}{3}$$
$$=27x^6\left(\frac{16}{x^4}\right)35$$
$$=15120x^2$$

So the coefficient of x^2 is 15120

(b) $\left(2x-\frac{2}{x^3}\right)^{10}$

General term is $(2x)^r\left(\frac{2}{x^3}\right)^{10-r}\binom{10}{r}$

Just looking at powers of x this is

$$x^r\left(\frac{2}{x^3}\right)^{10-r}$$
$$=x^r(x^{-3})^{10-r}$$
$$=x^rx^{3r-30}$$
$$=x^{4r-30}$$

In order to get the coefficient of x^2 we solve

$$4r-30=2$$
$$r=8$$

So the whole thing

$$(2x)^r \left(\frac{2}{x^3}\right)^{10-r} \binom{10}{r}$$

With $r = 8$

$$(2x)^8 \left(\frac{2}{x^3}\right)^{10-8} \binom{10}{8}$$

$$= 256x^8 \left(\frac{2}{x^3}\right)^2 45$$

$$= 46080x^2$$

So the coefficient of x^2 is 46080