Advanced Higher Homework 7
More Proof

Direct Proof

1. Using direct proof

   (a) Prove that if x is even then $x^2 + 3x$ is even

   (b) Prove that if x is rational then $x^2 + 3x$ is rational

Proof by Contradiction

2. Using proof by contradiction

   (a) Prove that there is no '*smallest fraction greater than 0*'

   (b) Prove that $\sqrt{7}$ is irrational

Proof by Induction

3. Prove by induction

   (a) $9^n + 3$ is divisible by 4 for all $n \geq 1$
   (b) $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for all $n \geq 0$
   (c) The recurrence relationship $u_{n+1} = u_n + 4n + 4, u_1 = 4$ has n^{th} term $2n(n+1)$

4. The Fibonacci numbers $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ have recurrence relationship $F_{n+1} = F_n + F_{n-1},\ n \geq 2.$

   Prove that

   (a) $\sum_{k=1}^{n} F_k = F_{n+2} - 1$
   (b) $F_n^2 = F_{n-1}F_{n+1} - 1$

Disproof by Counterexample

5. Find a counterexample to show each statement is sometimes false

   (a) If x and y are irrational, so is $x + y$

   (b) If x and y are rational, so is x^y

Proof by Contrapositive

6. Prove by contraposition

   (a) If $x^2 + 3x$ is even then x is even

   (b) If x is irrational then so is $2x$

Sequences and Series

7. A series has 2nd term 48 and 5th term 6.

   (a) Assuming it is an arithmetic series
   - find the 10th term and the sum of the first 10 terms

   (b) Assuming it is a geometric series
   - find the 10th term, and the sum of the first 10 terms, and the sum to infinity

Complex Numbers

8. For the complex number $z = 4 - 4i$ calculate

   (a) z^2

   (b) $|z|$

   (c) $|\bar{z} + i|$

   (d) $|\overline{z + i}|$

   (e) All five solutions to $\sqrt[5]{z}$

Advanced Higher Homework 7
More Proof Answers

Direct Proof

1. Using direct proof

(a) Prove that if x is even then $x^2 + 3x$ is even

If x is even then $x = 2k, k \in Z$

$$x^2 + 3x$$
$$= (2k)^2 + 3(2k)$$
$$= 4k^2 + 6k$$
$$= 2(2k^2 + 3)$$
$$= 2l, l \in Z$$

Hence $x^2 + 3x$ is even

(c) Prove that if x is rational then $x^2 + 3x$ is rational

If x is rational then $x = \frac{p}{q}, p \in Z, q \in Z$

$$x^2 + 3x$$
$$= \left(\frac{p}{q}\right)^2 + 3\left(\frac{p}{q}\right)$$
$$= \frac{p^2}{q^2} + \frac{3p}{q}$$
$$= \frac{p^2 + 3pq}{q^2}$$
$$= \frac{f}{g}, f \in Z, g \in Z$$

Hence $x^2 + 3x$ is rational

Proof by Contradiction

2. Using proof by contradiction

(a) Prove that there is no '*smallest fraction greater than 0*'

Assume for contradiction there is such a fraction, call it $\frac{a}{b}, a \in Z, b \in Z$

Then the fraction $\frac{a}{b+1}$ is smaller

This is a contradiction to it being the smallest fraction, hence the assumption of such a smallest fraction existing is false, and there is no smallest fraction greater than 0

(b) Prove that $\sqrt{7}$ is irrational

Assume for contradiction that $\sqrt{7}$ is rational, i.e. it can be written as $\frac{a}{b}, a \in Z, b \in Z$, with a and b having no common factors

Then

$$\sqrt{7} = \frac{a}{b}$$
$$7 = \frac{a^2}{b^2}$$
$$7b^2 = a^2$$

Since the left hand side is a multiple of 7 the right hand side must be too, which since 7 is prime is only possible if $a = 7c, c \in Z$

$$7b^2 = (7c)^2$$
$$7b^2 = 49c^2$$
$$b^2 = 7c^2$$

Then by similar logic to the above b is also a multiple of 7.
Hence a and b are both multiples of 7.
This violates the assumption that $\sqrt{7}$ is rational and so can be written as $\frac{a}{b}, a \in Z, b \in Z$, with a and b having no common factors, hence by contradiction $\sqrt{7}$ is irrational

Proof by Induction

3. Prove by induction

(a) $9^n + 3$ is divisible by 4 for all $n \geq 1$

Base case $n = 1$
$9^n + 3 = 12$ which is divisible by 4

Assume true for $n = k$ i.e. $9^k + 3$ is divisible by 4
Then

$$9^k + 3 = 4a, a \in Z$$
$$9(9^k + 3) = 9(4a)$$
$$9^{k+1} + 27 = 36a$$
$$9^{k+1} + 3 = 36a - 24$$
$$9^{k+1} + 3 = 4(9a - 6)$$

Hence also true for $n = k + 1$, so by induction true for all n

(b) $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for all $n \geq 0$

Base case $n = 0$

LHS $2^0 = 1$

RHS $2^{0+1-1} - 1 = 1$

Assume true for $n = p$ i.e. $\sum_{k=0}^{p} 2^k = 2^{p+1} - 1$

$$\sum_{k=0}^{p} 2^k = 2^{p+1} - 1$$

$$\sum_{k=0}^{p} 2^k + 2^{p+1} = 2^{p+1} - 1 + 2^{p+1}$$

$$\sum_{k=0}^{p+1} 2^k = 2^{p+2} - 1$$

Hence also true for $n = p + 1$, so by induction true for all n

(c) The recurrence relationship $u_{n+1} = u_n + 4n + 4, u_1 = 4$ has n^{th} term $2n(n + 1)$

Base case: $n = 1$

$2n(n + 1) = 2(1)(1 + 1) = 4$ so true

Assume true for $n = k$ so $u_k = 2k(k + 1)$

By recurrence rule

$$u_{k+1} = u_k + 4k + 4$$
$$u_{k+1} = (2k)(k + 1) + 4k + 4$$
$$u_{k+1} = (2k)(k + 1) + 4(k + 1)$$
$$u_{k+1} = 2(k + 1)(k + 2)$$
$$u_{k+1} = 2(k + 1)((k + 1) + 1)$$

Hence also true for $n = k + 1$, so by induction true for all n

4. The Fibonacci numbers $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ have recurrence relationship $F_{n+1} = F_n + F_{n-1},\ n \geq 2$.

Prove that

(a) $\sum_{k=1}^{n} F_k = F_{n+2} - 1$

Base case: $n = 1$

LHS: $F_1 = 1$

RHS: $F_3 - 1 = 2 - 1 = 1$

Assume true for $n = p$ so $\sum_{k=1}^{p} F_k = F_{p+2} - 1$

$$\sum_{k=1}^{p} F_k = F_{p+2} - 1$$

$$\sum_{k=1}^{p} F_k + F_{p+1} = F_{p+2} - 1 + F_{p+1}$$

$$\sum_{k=1}^{p+1} F_k = F_{p+2} - 1$$

(Using the definition of the Fibonacci Sequence)

Hence the also true for $n = p + 1$ hence true for all n

(b) $F_n^2 = F_{n-1}F_{n+1} - 1$

Base case $n = 2$

LHS $F_2^2 = 1^2 = 1$

RHS $F_1F_3 - 1 = 1(2) - 1 = 1$

Assume for induction true for $n = k$ so $F_k^2 = F_{k-1}F_{k+1} - 1$

Using the definition of the Fibonacci sequence $F_{k-1} + F_k = F_{k+1}$ so $F_{k-1} = F_{k+1} - F_k$

$$F_k^2 = F_{k-1}F_{k+1} - 1$$

$$F_k^2 = (F_{k+1} - F_k)F_{k+1} - 1$$

$$F_k^2 = F_{k+1}^2 - F_kF_{k+1} - 1$$

$$F_{k+1}^2 = F_kF_{k+1} - F_k^2 - 1$$

$$F_{k+1}^2 = F_k(F_{k+1} - F_k) - 1$$

$$F_{k+1}^2 = F_kF_{k+2} - 1$$

Using the definition of the Fibonacci sequence again

Hence the statement is true by induction for all n

Disproof by Counterexample

5. Find a counterexample to show each statement is sometimes false

(a) If x and y are irrational, so is $x + y$

One possibility is $x = \sqrt{2}$ and $y = 1 - \sqrt{2}$ in which case $x + y = 1$

(b) If x and y are rational, so is x^y

One possibility is $x = 2$ and $y = \frac{1}{2}$ in which case $x^y = 2^{\frac{1}{2}} = \sqrt{2}$

Proof by Contrapositive

6. Prove by contraposition

(a) If $x^2 + 4x$ is even then x is even

Start by assuming x is odd

$$x = 2k + 1$$
$$x^2 + 4x = (2k+1)^2 + 4(2k+1)$$
$$x^2 + 4x = 4k^2 + 4k + 1 + 8k + 4$$
$$x^2 + 4x = 4k^2 + 12k + 5$$
$$x^2 + 4k = 2(2k^2 + 6k + 2) + 1$$

Hence by assuming x is odd we have shown $x^2 + 4x$ is odd so by contrapositive if $x^2 + 4x$ is even then x is even

(b) If x is irrational then so is $2x$

Start by assuming $2x$ is rational i.e. $2x = \frac{a}{b}, a \in Z, b \in Z$

$$2x = \frac{a}{b}$$
$$x = \frac{a}{2b}$$

Hence x is also rational

So by contrapositive if x is irrational then so is $2x$

Sequences and Series

7. A series has 2nd term 48 and 5th term 6.

(a) Assuming it is an arithmetic series
- find the 10th term and the sum of the first 10 terms

$$u_2 = 48$$
$$a + d = 48$$
$$u_5 = 6$$
$$a + 4d = 6$$

Hence $a = 62, d = -14$

The tenth term is $a + 9d = 62 + 9(-14) = -64$

The sum of the first ten terms is

$$\frac{n}{2}(2a + (n-1)d)$$
$$= \frac{10}{2}\big(2(62) + (10-1)(-14)\big)$$
$$= -10$$

(b) Assuming it is a geometric series
- find the 10th term, and the sum of the first 10 terms, and the sum to infinity

$$u_2 = 48$$
$$ar^2 = 48$$
$$u_5 = 6$$
$$ar^5 = 6$$

Hence $a = 192, r = \frac{1}{2}$

The tenth term is $ar^9 = \frac{192}{512} = \frac{3}{8}$

The sum of the first ten terms is

$$\frac{a(1-r^{10})}{1-r}$$
$$= 192\frac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}}$$
$$= 383\frac{5}{8}$$

Complex Numbers

8. For the complex number $z = 4 - 4i$ calculate

(a) z^2

$$(4-4i)^2 = -32i$$

(b) $|z|$

$$|4-4i| = \sqrt{32} = 4\sqrt{2}$$

(c) $|\bar{z} + i|$

$$|4+4i+i| = \sqrt{41}$$

(d) $|\overline{z+i}|$

$$|4+3i| = 5$$

(e) All five solutions to $\sqrt[5]{z}$

To one decimal place (and omitting shortcutting the working here)

$$1.4 - 0.2i$$
$$0.6 + 1.3i$$
$$-1 + 1i$$
$$-1.3 - 0.6i$$
$$0.2 - 1.4i$$