Advanced Higher Homework 6
Proof (and complex numbers and Gaussian elimination)

Direct Proof

1. (a) Prove that the product of two odd numbers is an odd number

   (b) Prove that the product of two square numbers is a square number

Counterexample

2. Find a counter-example to disprove each claim:

   (a) All prime numbers are odd
   (b) If $x^2 > 25$ then $x > 5$
   (c) All square matrices have positive determinant

Induction

3. Prove by induction:

   (a) $2^{3n} - 1$ is a multiple of 7 for all $n \geq 0$

   (b) $11^n - 6$ is divisible by 5 for all $n \geq 1$

   (c) $2^n > n^2$ for all $n \geq 5$

   (d) The sum of the first n even numbers is $n^2 + n$

Complex Numbers

4. Find the square root(s) of $-48 + 14i$

5. For the function $f(z) = z^4 - 4z^3 + 17z^2 - 16z + 52$

   (a) Show that $2i$ is a root, and find another root
   (b) Hence fully factorize $f(z)$

6. Use DeMoivre's theorem to find all the solutions of $z^3 = 27i$

Gaussian Elimination

7. Solve by Gaussian elimination:

$$3a + 2b + 4c = 10, \qquad a - b - c = 5, \qquad 7a + 3b + 7c = 25$$

Partial Fractions

8. For the function $f(x) = x^3 + 5x^2 + 9x + 5$

   (a) Show that $(x - 1)$ is a factor and hence fully factorise $f(x)$

   (b) Express $\frac{x^4+7x}{x^3+5x^2+9x+5}$ in the form $A + Bx + \frac{Cx^2+Dx+E}{x^3+5x^2+9x+5}$

   (c) Hence write $\frac{x^4+7x}{x^3+5x^2+9x+5}$ in partial fractions

Advanced Higher Homework 6 – Answers

Proof (and Complex Numbers, Gaussian elimination, Partial Fractions)

Direct Proof

1. Use direct proof to show that

(a) The product of two odd numbers is an odd number

Let the first odd number be $2x + 1$ and the second $2y + 1$

The product is

$$(2x+1)(2y+1)$$
$$= 4xy + 2x + 2y + 1$$
$$= 2(xy + x + y) + 1$$

Which is also an odd number

(b) The product of two square numbers is a square number

Let the first square number be x^2 and the second square number be y^2

The product is

$$x^2y^2$$
$$= (xy)^2$$

Which is also a square number

(c) Every month contains at least four Tuesdays

Every month has at least 28 days

One of the first 7 days must be a Tuesday

If the 1st day of the month is a Tuesday then so are the 8th, 15th, 22nd

If the 2nd day of the month is a Tuesday then so are the 9th, 16th, 23rd

If the 3rd day ….

If the 4th day …

If the 5th day ….

If the 6th day ….

If the 7th day of the month is a Tuesday then so are the 14th, 21st, 28th

Counterexample

2. Find a counter-example to disprove each claim:

(a) All prime numbers are odd

2 is a prime number and is not odd

(b) If $x^2 > 25$ then $x > 5$

$x = -10$, if $x^2 > 25$ but it is not true that $x > 5$

(c) All matrices have positive determinant

For example the simple 1 by 1 matrix (-2) does not have positive determinant

Proof by Induction

3. Prove by induction:

(a) $2^{3n} - 1$ is a multiple of 7 for all $n \geq 0$

Base case with $n = 1$

$2^{3(1)} - 1 = 7$ which is a multiple of 7

Moivre.

Assume true for $n = k$

$$2^{3k} - 1 = 7a$$

Need to show true for $n = k + 1$

$$2^{3(k+1)} - 1 = 7b$$

Start with $n = k$ line:

$$2^{3k} - 1 = 7a$$
$$2^{3k} = 7a + 1$$
$$2^3 \times 2^{3k} = 2^3(7a + 1)$$
$$2^{3k+3} = 56a + 8$$
$$2^{3(k+1)} - 1 = 56a + 7$$
$$2^{3(k+1)} - 1 = 7(8a + 1)$$

We have shown the conjecture holds for $n = 1$ and since (true for $n = k$) implies (true for $n = k + 1$) then by induction it is true for all n

(b) $11^n - 6$ is divisible by 5 for all $n \geq 1$

Base case with $n = 1$

$$11^1 - 6 = 5$$

Assume true for $n = k$

$$11^k - 6 = 5a$$

Need to show true for $n = k + 1$

$$11^{k+1} - 6 = 5b$$

Start with $n = k$ line

$$11^k - 6 = 5a$$
$$11^k = 5a + 6$$
$$11^{k+1} = 11(5a + 6)$$
$$11^{k+1} = 55a + 66$$
$$11^{k+1} - 6 = 55a + 60$$
$$11^{k+1} - 6 = 5(11a + 12)$$

We have shown the conjecture holds for $n = 1$ and since (true for $n = k$) implies (true for $n = k + 1$) then by induction it is true for all n

(c) $2^n > n^2$ for all $n \geq 5$

Base case $n = 5$
LHS: 32, RHS: 25 so is true

Assume true for $n = k$

$$2^k > k^2$$

Show true for $n = k + 1$

$$2^{(k+1)} > (k + 1)^2$$

Start with $n = k$ line:

$$2^k > k^2$$
$$2 \times 2^k > 2k^2$$
$$2^{k+1} > k^2 + k^2$$

Since $k > 5$ it is true that $k^2 > 2k + 1$
(this could be proved by a separate proof by induction but is reasonably obvious so we just state it as a fact here)

$$2^{k+1} > k^2 + k^2 > k^2 + 2k + 1$$
$$2^{k+1} > (k + 1)^2$$

We have shown the conjecture holds for $n = 5$ and since (true for $n = k$) implies (true for $n = k + 1$) then by induction it is true for all $n \geq 5$

(d) The sum of the first n even numbers is $n^2 + n$

Base case $n = 1$
LHS 2
RHS $1^2 + 1 =$

Assume true for $n = k$
Sum of first k even numbers is $k^2 + k$

Show true for $n = k + 1$
Sum of first $k + 1$ even numbers is $(k + 1)^2 + (k + 1)$

Start with $n = k$ line

$$\sum_{1}^{k} 2r = k^2 + k$$
$$\sum_{1}^{k} 2r + 2(k + 1) = k^2 + k + 2(k + 1)$$

$$\sum_{1}^{k+1} 2r = k^2 + 3k + 2$$
$$\sum_{1}^{k+1} 2r = (k+1)^2 + k + 1$$

We have shown the conjecture holds for $n = 5$ and since (true for $n = k$) implies (true for $n = k + 1$) then by induction it is true for all $n \geq 1$

(Note that this proof is just the same as the standard $\sum_1^n k = \frac{1}{2}n(n+1)$ but with bot sides doubled)

Complex Numbers

4. Find the square root(s) of $-48 + 14i$

   The solutions are $\pm(1 + 7i)$

   (for the method see previous solutions)

5. For the function $f(z) = z^4 - 4z^3 + 17z^2 - 16z + 52$

   (a) Show that $2i$ is a root, and find another root

   $$f(2i) = 0$$

   The complex conjugate is also a root, hence another root is $-2i$

   (b) Hence fully factorize $f(z)$

   Two factors are $2i$ and $-2i$.
   Hence two factors are $(z - 2i)$ and $(z + 2i)$ so the product $(z + 2i)(z - 2i) = z^2 + 4$ is also a factor

   By inspection,
   $$f(z) = (z^2 + 4)(z^2 - 4z + 13)$$

   Factorising the quadratics gives the full factorisation
   $$f(z) = (z - 2i)(z + 2i)(z - 2 - 3i)(z - 2 + 3i)$$

6. Use DeMoivre's theorem to find all the solutions of $z^3 = 27i$

   Writing the RHS in polar form
   $$27i = 27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Back to the original equation

$$z^3 = 27i$$
$$z^3 = 27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$
$$z = 27^{\frac{1}{3}}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\frac{1}{3}}$$
$$z = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\frac{1}{3}}$$

Applying DeMoivre's Theorem

$$z_1 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$z_1 = \frac{3}{2}\sqrt{3} + \frac{3}{2}\mathrm{i}$$

Rewriting z with intervals of 2π

$$z = 3\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)^{\frac{1}{3}}$$
$$z_2 = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$
$$z_2 = -\frac{3}{2}\sqrt{3} + \frac{3}{2}\mathrm{i}$$

$$z = 3\left(\cos\frac{9\pi}{2} + i\sin\frac{9\pi}{2}\right)^{\frac{1}{3}}$$
$$z_3 = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$
$$z_3 = -3i$$

Gaussian Elimination

7. Solve by Gaussian elimination:

$$3a + 2b + 4c = 10$$
$$a - b - c = 5$$
$$7a + 3b + 7c = 25$$

Set up augmented matrix and apply EROs
Matrix is indeterminate, with general solutions

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -0.4 \\ -1.4 \\ 1 \end{pmatrix}$$

Partial Fractions

8. For the function $f(x) = x^3 + 5x^2 + 9x + 5$

(a) Show that $(x+1)$ is a factor and hence fully factorise $f(x)$

$f(-1) = 0$ hence $(x+1)$ is a factor

Factorising by inspection gives $f(x) = (x+1)(x^2+4x+5)$

(b) Express $\frac{x^4+7x}{x^3+5x^2+9x+5}$ in the form $A + Bx + \frac{Cx^2+Dx+E}{x^3+5x^2+9x+5}$

$$\frac{x^4+7x}{x^3+5x^2+9x+5} = x - 5 + \frac{16x^2+47x+25}{x^3+5x^2+9x+5}$$

(c) Hence express $\frac{x^4+7x}{x^3+5x^2+9x+5}$ in partial fractions

Note that $x^2 + 4x + 5$ has determinant -4 so is irreducible (over the reals)

$$\frac{x^4+7x}{x^3+5x^2+9x+5} = x - 5 + \frac{A}{x+1} + \frac{Bx+C}{x^2+4x+5}$$

$$\frac{x^4+7x}{x^3+5x^2+9x+5} = x - 5 - \frac{3}{x+1} + \frac{19x+40}{x^2+4x+5}$$