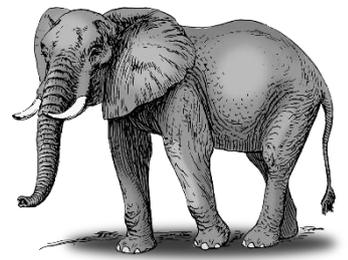


Advanced Higher Homework 5

Proof by Induction & Curve Sketching



Proof by Induction

1. Prove by induction

(a) $n^7 - n$ is divisible by 7 for all $n \geq 1$

(b) $n^2 < 3^n$ for all $n \geq 1$

(hint: you can use the fact that $2k^2 > 2k + 1$ for $k \geq 2$)

(c) The recurrence relationship $u_{n+1} = u_n + 2n + 2, u_1 = 1$ has n^{th} term $n^2 + n - 1$

(hint: assume for induction that $u_k = k^2 + k - 1$ and show $u_{k+1} = (k + 1)^2 + (k + 1) - 1$)

(d) $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers $n \geq 1$

Curve Sketching

2. For the function

$$f(x) = \frac{x^2 + 1}{x^2 - 1}$$

- Write the function in partial fractions (doing an algebraic division first)
- Find $f(0)$
- For which values of x is the function undefined?
- What does the value of the function tend to as x approaches infinity?
- What does the value of the function tend to as x approaches minus infinity?
- Find the roots, if any
- Differentiate the function to find any turning points
- Sketch the function, marking on all the key features

3. Sketch the function

$$f(x) = \frac{x^2 - 3x + 1}{x - 3}$$

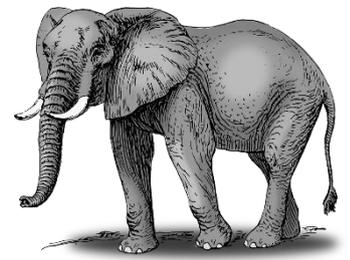
Sequences and Series

4. A series has 5th term 160 and 10th term 5.
- (a) Assuming it is an arithmetic series
 - find the 20th term and the sum of the first 20 terms
 - (b) Assuming it is a geometric series
 - find the 20th term, and the sum of the first 20 terms, and the sum to infinity

Complex Numbers

5. For the complex number $z = -7 + 4\sqrt{2}i$ calculate
- (a) z^2
 - (b) $|z|$
 - (c) All four solutions to $\sqrt[4]{z}$, by first finding \sqrt{z}
 - (d) Plot the four solutions on an Argand diagram

Advanced Higher Homework 5
Proof by Induction & Curve Sketching
Answers



Proof by Induction

1. Prove by induction

(a) $n^7 - n$ is divisible by 7 for all $n \geq 1$

Base case: $n = 1$

$$1^7 - 1 = 0$$

Which is divisible by 7

Assume for induction true for $n = k$:

$$k^7 - k = 7a$$

Then for case $n = k + 1$

$$\begin{aligned} & (k + 1)^7 - (k + 1) \\ &= [k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1] - (k + 1) \\ &= k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 6k \\ &= [k^7 - k] + [7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k] \\ &= 7a + 7[k^6 + 3k^5 + 5k^4 + 5k^3 + 3k^2 + k] \text{ (using result for } n = k) \\ &= 7a + 7b \end{aligned}$$

Which is also a multiple of 7

(b) $n^2 < 3^n$ for all $n \geq 1$

Base Case: $n = 1$

$$\text{LHS } 1^2 = 1$$

$$\text{RHS } 3^1 = 3 \text{ so true}$$

Assume for induction true for $n = k$:

$$k^2 < 3^k$$

Then for case $n = k + 1$

$$\begin{aligned} & (k + 1)^2 \\ &= k^2 + 2k + 1 \\ &< 3^k + 2k + 1 \text{ (using result for } n = k) \\ &< 3^k + 3^k \text{ (using the fact that } 3^k > k^2 > 2k + 1 \text{ for } k \geq 2) \\ &< 3^{k+1} \end{aligned}$$

So also true for $n = k + 1$

(c) The recurrence relationship $u_{n+1} = u_n + 2n + 2, u_1 = 1$ has n^{th} term $n^2 + n - 1$

Base case: $n = 1$

$$n^2 + n - 1 = 1^2 + 1 - 1 = 1 \text{ so true}$$

Assume true for $n = k$ so $u_k = k^2 + k - 1$

By recurrence rule

$$u_{k+1} = u_k + 2k + 2$$

$$u_{k+1} = k^2 + k - 1 + 2k + 2$$

$$u_{k+1} = k^2 + 3k + 1$$

$$u_{k+1} = (k + 1)^2 + (k + 1) - 1$$

Hence also true for $n = k + 1$, so by induction true for all n

(d) $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers $n \geq 1$

Base case: $n = 1$

$$LHS = \frac{d}{dx}(x^1) = 1$$

$$RHS = 1x^0 = 1$$

Assume true for $n = k$

$$\frac{d}{dx}(x^k) = kx^{k-1}$$

Then start with LHS true for $n = k + 1$ and use the chain rule

$$\begin{aligned} & \frac{d}{dx}(x^{k+1}) \\ &= \frac{d}{dx}(x^k x) \\ &= x^k \frac{d}{dx}(x) + x \frac{d}{dx}(x^k) \\ &= x^k + x k x^{k-1} \\ &= x^k + k x^k \\ &= (k + 1)x^k \\ &= (k + 1)x^{(k+1)-1} \end{aligned}$$

Hence also true for $n = k + 1$, so by induction true for all n

Curve Sketching

2. For the function

$$f(x) = \frac{x^2 + 1}{x^2 - 1}$$

(a) Write the function in partial fractions (doing an algebraic division first)

$$f(x) = 1 - \frac{1}{x+1} + \frac{1}{x-1}$$

(b) Find $f(0)$

$$f(0) = -1$$

(c) For which values of x is the function undefined?

$$x = -1 \text{ or } x = 1$$

(d) What does the value of the function tend to as x approaches infinity?

Approaches 1 (from above)

(e) What does the value of the function tend to as x approaches minus infinity?

Approaches 1 (from above)

(f) Find the roots, if any

From considering

$$f(x) = \frac{x^2 + 1}{x^2 - 1}$$

There are no solutions to $f(x) = 0$ because $x^2 + 1 \geq 0$ for all x

Hence there are no roots

(g) Differentiate the function to find any turning points

$$f'(x) = -\frac{4x}{(x^2 - 1)^2}$$

This is 0, when $x = 0$, which means $f(x) = -1$

Hence a turning point is at $(0, -1)$

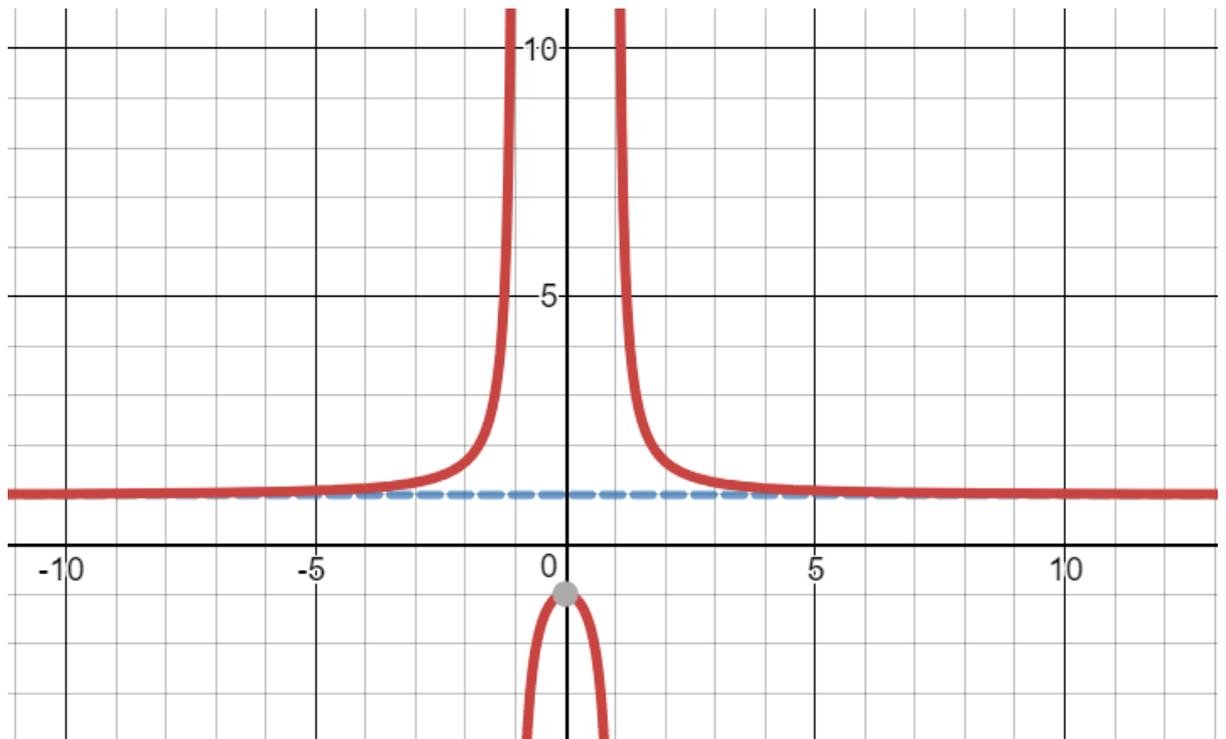
To find the nature of the turning point, find the second derivative (or use a nature table)

$$f''(x) = \frac{4(3x^2 + 1)}{(x^2 - 1)^3}$$

This is negative at $x = 0$ so the turning point is a maximum

(h) Sketch the function, marking on all the key features

Knowing the function has two vertical asymptotes at $x=1$ and $x=-1$, a horizontal asymptote at $y=1$, a maximum at $(0,-1)$, gives nearly the whole function. Exploring a few other values of x (e.g. $x=1$, $x=-1$) gives the whole graph:



3. Sketch the function

$$f(x) = \frac{x^2 - 3x + 1}{x - 3}$$

First use algebraic division to get

$$f(x) = x + \frac{1}{x - 3}$$

This crosses the axis at

$$f(0) = -\frac{1}{3}$$

The function is undefined for $x = 3$

As x approaches infinity it becomes $y = x$

As x approaches minus infinity it becomes

The roots occur when

$$f(x) = x + \frac{1}{x - 3} = 0$$

Solving this equation $x + \frac{1}{x - 3} = 0$ is a quadratic, with solutions $x = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Differentiating gives

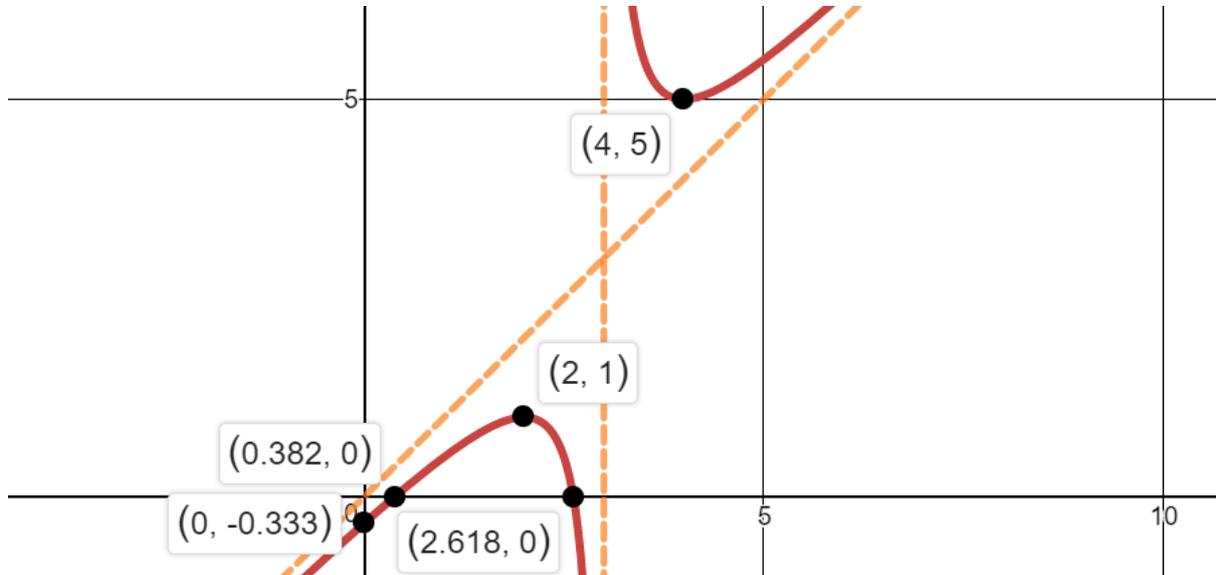
$$f'(x) = 1 - \frac{1}{(x - 3)^2}$$

This is zero when

- $x = 2$ (a maximum, from differentiating again or nature table), where $f(x) = 1$
- $x = 4$ (a minimum, from differentiating again or nature table), where $f(x) = 5$

It is positive when $x < 2$ or $x > 4$ else negative

Hence the full graph is



Sequences and Series

4. A series has 5th term 160 and 10th term 5.

(a) Assuming it is an arithmetic series

- find the 20th term and the sum of the first 20 terms

$$\begin{aligned} u_5 &= 160 \\ a + 4d &= 160 \\ u_{10} &= 5 \\ a + 9d &= 5 \end{aligned}$$

Hence $a = 284, d = -31$

The twentieth term is $a + 19d = 284 + 19(-31) = -305$

The sum of the first twenty terms is

$$\begin{aligned} &\frac{n}{2}(2a + (n-1)d) \\ &= \frac{20}{2}(2(284) + (20-1)(-31)) \\ &= -210 \end{aligned}$$

(b) Assuming it is a geometric series

- find the 20th term, and the sum of the first 20 terms, and the sum to infinity

$$\begin{aligned} ar^4 &= 160 \\ ar^9 &= 5 \end{aligned}$$

Hence $a = 2560, r = \frac{1}{2}$

The twentieth term is $ar^{19} = \frac{5}{1024}$

The sum of the first ten terms is

$$\begin{aligned} & \frac{a(1 - r^{20})}{1 - r} \\ &= 2560 \frac{1 - \left(\frac{1}{2}\right)^{20}}{1 - \frac{1}{2}} \\ &= 5019 \frac{523}{524} \end{aligned}$$

Complex Numbers

5. For the complex number $z = -7 + 4\sqrt{2}i$ calculate

(a) $z^2 = 17 - 56\sqrt{2}i$

(b) $|z| = 9$

(c) All four solutions to $\sqrt[4]{z}$, by first finding \sqrt{z}

$$z_1 = \sqrt{2} + i$$

$$z_2 = -1 + \sqrt{2}i$$

$$z_3 = -\sqrt{2} - i$$

$$z_4 = 1 - \sqrt{2}i$$

(d) Plot the solutions on an Argand diagram

