Sequences & Series
Arithmetic Seq. & Ser.

QUESTION

1. Find the formula for the n^{th} term of each of the following sequences and find the requested term.
   (a) 3, 11, 19,.......... ; u_{19}
   (b) 8, 5, 2, ... ; u_{15}
   (c) 7, 6·5, 6,........... ; u_{12}

2. Find the number of terms in each of the following sequences.
   (a) 2, 4, 6, ..., 46.
   (b) 50, 47, 44, ..., 14.
   (c) 2, –9, –20, ..., –130.

3. State the values of a and d in each of the following series and find the requested S_n.
   (a) 4 + 10 + 16 +.........., S_{12}.
   (b) 15 + 13 + 11 +........, S_{20}.
   (c) 20 + 13 + 6 +.........., S_{16}.

4. For each of the Arithmetic Sequences, find S_n indicated. (Find a and d first).
   (a) $u_2 = 15$, $u_5 = 21$, S_{10}.
   (b) $u_4 = 18$, d = –5, S_{16}.
   (c) $u_3 = 7$, $u_{12} = 61$, S_{15}.

5. $S_{10} = 120$, $S_{20} = 840$, find S_{30}.

6. $u_{15} = 7$, $S_9 = 18$, find a, d and u_{20}.

7. How many terms of the Arithmetic series
   28 + 24 + 20 + ...
   does it take to give a sum of zero ?

8. The sixth term of an Arithmetic sequence is twice the third term.
   If the first term is 3, find d and the tenth term.

9. How many terms of the Arithmetic series
   1 + 3 + 5 +........ will give a sum of 1521 ?

Sequences & Series
Arithmetic Seq. & Ser.

ANSWERS

1. (a) $u_n = 8n - 5,\ u_{19} = 147$ (b) $u_n = 11 - 3n,\ u_{15} = -34$
   (c) $u_n = 7\cdot 5 - 0\cdot 5n,\ u_{12} = 1\cdot 5$
2. (a) $n = 23$ (b) $n = 13$ (c) $n = 13$
3. (a) $a = 4,\ d = 6,\ S_{12} = 444$ (b) $a = 15, d = -2,\ S_{20} = -80$
   (c) $a = 20,\ d = -7,\ S_{16} = -520$
4. (a) $a = 13,\ d = 2,\ S_{10} = 220$ (b) $a = 133, d = -5,\ S_{16} = -72$
   (c) $a = -5,\ d = 6,\ S_{15} = 555$
5. $a = -15,\ d = 6,\ S_{30} = 2160$
6. $a = 0,\ d = {}^1/_2,\ u_{20} = 9{\cdot}5$
7. $n = 0$ or 15
8. $d = 3,\ u_{10} = 30$
9. 39 terms

Sequences & Series
Finite Geometric Seq. & Ser.

QUESTION 1

1. Find the common ratio for each of these Geometric Series.

(a) 1, 3, 9, 27, ... (b) 12, 6, 3, 1·5, ...

(c) 7, 0·7, 0·07, ... (d) 18, 54, 162, ...

(e) 2·25, 1·5, 1, ... (f) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots$

(g) 1, −1, 1, −1, ... (h) 1, −2, 4, −8, ...

2. Write down the first 4 terms of these Geometric Sequences :-

(a) $u_n = 3^{(n-1)}$ (b) $u_n = 3(-2)^{(n-1)}$

(c) $u_n = 6\left(\frac{1}{2}\right)^{(n-1)}$

3. Find the required term in the following Geometric Sequences. (Find u_n first)

(a) 1, 2, 4,... ; u_5 (b) 2, 6, 18,... ; u_6

(c) 4, 12, 36,... ; u_6 (d) 2, 20, 200,... ; u_5

(e) 1, −2, 4,... ; u_6 (f) 6, 3, $^3/_2$,... ; u_7

4. Find the formula for the n^{th} term of these Geometric Sequences :- (i.e. find u_n).

(a) 1, 2, 4, ... (b) 3, 6, 12, ...

(c) 2, −6, 18, ... (d) 9, 3, 1, ...

(e) 4, 2, 1, ... (f) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

5. Find the common ratio and the 5th term of these Geometric Sequences :-

(a) $a = 6,\ u_3 = 24$ (b) $a = 50,\ u_4 = 400$

(c) $a = 36,\ u_2 = -12$

6. Find the sum of each of the following Geometric Series and simplify the answer as far as possible.

(a) $1 + 2 + 4 + \ldots$ to 8 terms (b) $2 + 6 + 18 + \ldots$ to 6 terms.

(c) $2 - 4 + 8 - \ldots$ to 5 terms (d) $2 - 6 + 18 - \ldots$ to 5 terms

(e) $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ to 6 terms (f) $1 + \frac{1}{3} + \frac{1}{9} + \ldots$ to 5 terms

(g) $1 + x + x^2 + \ldots$ to n terms (h) $1 - y + y^2 - \ldots$ to n terms.

7. Find n if :-

(a) $3 + 3^2 + 3^3 + 3^4 + \ldots. + 3^n = 363$ (b) $2 + 2^2 + 2^3 + 2^4 + \ldots\ldots. + 2^n = 510$

Sequences & Series
Finite Geometric Seq. & Ser.

ANSWERS

1. (a) $r = 3$ (b) $r = 1/2$ (c) $r = 1/10$ (d) $r = 3$
   (e) $r = 2/3$ (f) $r = 1/2$ (g) $r = -1$ (h) $r = -2$

2. (a) 1, 3, 9, 27 (b) 3, –6, 12, –24, (c) 6, 3, $3/2$, $3/4$,

3. (a) $u_n = 2^{n-1},\ u_5 = 16$ (b) $2 \times 3^{n-1},\ u_6 = 486$
   (c) $u_n = 4 \times 3^{n-1},\ u_6 = 972$ (d) $2 \times 10^{n-1},\ u_5 = 20000$
   (e) $u_n = (-2)^{n-1},\ u_6 = -32$ (f) $6 \times \left(\frac{1}{2}\right)^{n-1},\ u_7 = \frac{3}{32}$

4. (a) $u_n = 2^{n-1}$ (b) $u_n = 3 \times 2^{n-1}$
   (c) $u_n = 2 \times (-3)^{n-1}$ (d) $u_n = 9 \times \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{n-3}$
   (e) $u_n = 4 \times \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n-3}$ (f) $u_n = \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n}$

5. (a) $r = \pm 2,\ u_5 = 96$ (b) $r = 2,\ u_5 = 800$ (c) $r = -\frac{1}{3},\ u_5 = \frac{4}{9}$

6. (a) 255 (b) 728 (c) 22 (d) 122
   (e) $1\frac{31}{32}$ (f) $1\frac{41}{80}$ (g) $\frac{1 - x^n}{1 - x}$ (h) $\frac{1 - (-y)^n}{1 + y}$

7. (a) $n = 5$ (b) $n = 8$

Sequences & Series
Infinite Geometric Seq. & Ser.

QUESTION

Find the common ratio and hence state whether the sum to infinity exists.
If the sum to infinity exists find it.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \ldots$

(b) $1 + 2 + 4 + \ldots$

(c) $4 + 1 + \frac{1}{4} + \ldots$

(d) $8 + 4 + 2 + \ldots$

(e) $1 - 5 + 25 - \ldots$

(f) $10 - 9 + 8{\cdot}1 - .$

(g) $1 - \frac{1}{2} + \frac{1}{4} - \ldots$

(h) $2 + \frac{4}{3} + \frac{8}{9} + \ldots$

ANSWERS

(a) $r = \frac{1}{3}$, $S_\infty = 1\frac{1}{2}$

(b) $r = 2$, S_∞ does not exist

(c) $r = \frac{1}{4}$, $S_\infty = 5\frac{1}{3}$

(d) $r = \frac{1}{2}$, $S_\infty = 16$

(e) $r = -5$, S_∞ does not exist

(f) $r = -\frac{9}{10}$, $S_\infty = \frac{100}{19}$

(g) $r = -\frac{1}{2}$, $S_\infty = \frac{2}{3}$

(h) $r = \frac{2}{3}$, $S_\infty = 6$

Sequences & Series
Infinite Geometric Seq. & Ser.

QUESTIONS

1. Expand the following in ascending powers of x, giving the first 4 terms.

(a) $\dfrac{1}{1+2x}$ (b) $\dfrac{1}{1-3x}$ (c) $\dfrac{1}{1+\frac{x}{2}}$

2. Expand the following in ascending powers of x giving the first 4 terms.

(a) $\dfrac{1}{2+4x}$ (b) $\dfrac{1}{3-x}$ (c) $\dfrac{1}{2-3x}$

3 ~~4~~. Expand $\dfrac{1-3x}{1+4x}$ in ascending powers of x as far as x^3. $\left((1-3x)\times\dfrac{1}{(1+4x)}\right)$.

ANSWERS

1. (a) $\dfrac{1}{1+2x}=1-2x+4x^2-8x^3+\dots$ (b) $\dfrac{1}{1-3x}=1+3x+9x^2+27x^3+\dots$

(c) $\dfrac{1}{1+\frac{x}{2}}=1-\dfrac{x}{2}+\dfrac{x^2}{4}-\dfrac{x^3}{8}+\dots$

2. (a) $\dfrac{1}{2+4x}=\dfrac{1}{2(1+2x)}=\dfrac{1}{2}-x+2x^2-4x^3+\dots$

(b) $\dfrac{1}{3-x}=\dfrac{1}{3\left(1-\frac{1}{3}x\right)}=\dfrac{1}{3}+\dfrac{x}{9}+\dfrac{x^2}{27}+\dfrac{x^3}{81}+\dots$

(c) $\dfrac{1}{2-3x}=\dfrac{1}{2\left(1-\frac{3}{2}x\right)}=\dfrac{1}{2}+\dfrac{3x}{4}+\dfrac{9x^2}{8}+\dfrac{27x^3}{16}+\dots$

3 ~~4~~. $\dfrac{1-3x}{1+4x}=1-7x+28x^2-112x^3+\dots$

Sequences & Series
Sigma Notation & Series

QUESTION

Evaluate:-

1. $\sum_{k=1}^{10} k$ 2. $\sum_{k=1}^{20} 2k$ 3. $\sum_{k=1}^{8} (2k+3)$

4. $\sum_{k=1}^{20} (4k+5)$ 5. $\sum_{k=1}^{10} (3k-1)$

ANSWERS

1. 55 2. 420 3. 96 4. 940 5. 155

QUESTION

1. Simplify (i.e. express in terms of n)

(a) $\sum_{r=1}^{n} 3r+2$ (b) $\sum_{r=1}^{n} 5-2r$ (c) $\sum_{r=1}^{n} r^2-r$ (d) $\sum_{r=1}^{n} r(r^2-2)$

2. Express each of these sums in terms of sigma notation and then simplify

(a) $2^2+4^2+6^2+8^2+...+(2n)^2$

(b) $1\times 4+4\times 7+7\times 10+10\times 13+...+(3n-2)\times(3n+1)$

3. Building blocks are piled in a triangular pyramid so that the numbers in successive layers from the top down are 1, $1+2$, $1+2+3$, $1+2+3+4$ ... . Find the number of blocks in a pyramid with n layers.

HINT: First of all find the nth term.

4. By obtaining a formula, in fully factorised form, for $\sum_{k=1}^{n} 2k^3+k^2-k$ in terms of n evaluate $(2+1-1)+(16+4-2)+.....+(2000+100-10)$.

5. Express $\dfrac{1}{(2x-1)(2x+1)}$ in partial fractions. Hence deduce that

$$\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$

Sequences & Series
Sigma Notation & Series

ANSWERS

1. (a) $\frac{n}{2}(3n+7)$ (b) $4n-n^2$ (c) $\frac{n}{3}(n+1)(n-1)$ (d) $\frac{n}{4}(n+1)(n^2+n-4)$

2. (a) $\sum_{k=1}^{n}(2k)^2=\sum_{k=1}^{n}4k^2=\frac{2n}{3}(n+1)(2n+1)$

   (b) $\sum_{k=1}^{n}(3k-2)(3k+1)=n(3n^2+3n-2)$

3. nth row has $\frac{n^2}{2}+\frac{n}{2}$ bricks in it. Therefore the total number of bricks is

   $\sum_{r=1}^{n}\frac{r^2}{2}+\frac{r}{2}=\frac{n(n+1)(n+2)}{6}$.

4. $\frac{n}{6}(n+1)(n+2)(3n-1)$. When $n=10$ we get 6380.

5. $\frac{1}{2(2x-1)}-\frac{1}{2(2x+1)}$. Prove result via "telescopic series" method.

Sequences & Series
Maclaurin Series

QUESTION 1

Expand the following functions in ascending powers of x as far as the power indicated.

1. $f(x) = \cos x$ as far as x^6.
2. $f(x) = \tan x$ as far as x^3.
3. $f(x) = \sin^{-1} x$ as far as x^3.
4. $f(x) = \ln(1 - x)$ as far as x^4.
5. $f(x) = e^{3x}$ as far as x^4.
6. $f(x) = \ln(1 + 2x)$ as far as x^5.
7. $f(x) = \sin 3x$ as far as x^5.
8. $f(x) = \tan 2x$ as far as x^5.
9. $f(x) = \ln(2 + x)$ as far as x^3. (Hint: $\ln(2 + x) = \ln 2(1 + \frac{x}{2})$)

ANSWER 1

1. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$
2. $x + \frac{1}{3}x^3$
3. $x + \frac{1}{6}x^3$
4. $-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$
5. $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$
6. $2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \frac{32}{5}x^5$
7. $3x - \frac{9}{2}x^3 + \frac{81}{40}x^5$
8. $2x + \frac{8}{3}x^3 + \frac{64}{15}x^5$
9. $\ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3$

QUESTION

1. Expand $e^{\sin x}$ as far as the term in x^4.
2. Expand $\ln(1 + \sin x)$ as far as the term in x^4.
3. Expand $e^x \sin x$ as far as the term in x^5.
4. Expand $\ln(1 + e^x)$ as far as the term in x^4.

Sequences & Series
Maclaurin Series

ANSWERS

1. $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4$

2. $x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4$

3. $x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5$

4. $\ln 2+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{192}x^4$

Sequences & Series
Maclaurin Series

QUESTIONS

(1) Find the term in x^4 in the expansion of $e^x(1 + 2x)$.

(2) Find the coefficient of x^6 in the expansion of $\dfrac{3-4x-x^2}{e^x}$.

(3) Find the term in x^5 in the expansion of $e^x(1 - 3x)$.

(4) Find the coefficient of x^4 in the expansion of $(e^x - 1)(x^2 - 2x - 1)$.

(5) Write down the power series for $\ln(1 - x)$.

By writing $1 + x + x^2$ in the form $\dfrac{1-x^3}{1-x}$, show that for $-1 \le x \le 1$,

$$\ln(1 + x + x^2) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{3}x^6 + \ldots\ldots$$

(6) Expand $\ln(1 + \sin x)$ as far as the term in x^4.

(7) a) If $f(x) = \cos^2 x$, show that $f'(x) = -\sin 2x$. By finding higher derivatives obtain the Maclaurin series for $\cos^2 x$ up to the term x^6.

b) By writing $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and replacing $\cos 2x$ by a suitable power series obtain the expansion of $\cos^2 x$ up to the term x^6.

(8) Expand $e^{\sin x}$ as far as the term in x^3.

(9) Expand e^{2x+3} as far as the term in x^4.

(10) Express $(1 + x)^{1+x}$ in the form $e^{f(x)}$ and deduce the result

$$(1 + x)^{1+x} = 1 + x + x^2 + \frac{1}{2}x^3 + \ldots.$$

(11) Expand $\dfrac{x}{e^x - 1}$ as far as the term in x^4.

(12) Show that $\ln(1 + e^x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots.$

(13) Find the power series for $\ln(1 - x - 2x^2)$ as far as x^5. Give the domain of validity.

(14) By expanding $(1 + x)^{-1}$ and integrating the resulting series, show that

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots.$$

Sequences & Series
Maclaurin Series

Answers

(1) $\frac{9x^4}{24}$ (2) $\frac{19x^6}{240}$ (3) $-\frac{7x^5}{60}$ (4) $\frac{x^4}{8}$

(5) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

(6) $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{6}$

(7) (a) $\cos^2 x = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45}$

(b) $\cos^2 x = \frac{1}{2}(1+\cos 2x)\ 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45}$

(8) $1 + x + \frac{1}{12}x^2 - \frac{5}{36}x^3$

(9) $38\frac{7}{8} + 55x + 27x^2 + \frac{16}{3}x^3 + \frac{2}{3}x^4$

(10) Proof

(11) $\frac{120}{120 + 60x + 20x^2 + 5x^3 + x^4}$

(12) Proof

(13) $-1 < x < -0.375;\ -x - \frac{5}{2}x^2 - \frac{7}{3}x^3 - 4x^4 - 4x^5$

(14) Proof