Proof, Logic & Number Theory
Logic & Proof

QUESTION

Assuming the variables are on the set of real numbers, which of the following implications are true or false.

1. If $x = 5$ then $x^2 = 25$.

2. If $a > 0$ then $a^2 > 0$.

3. If $ax^2 + bx + c = 0$ has real roots then $b^2 - 4ac < 0$.

4. If $f'(a) = 0$ then $f(a)$ is a maximum stationary value of f.

5. If $\underline{u}.\underline{v} = 0$ and $|\underline{u}| \neq 0$ and $|\underline{v}| \neq 0$, then $\underline{u}$ is perpendicular to $\underline{v}$.

6. If $g(x) = x^2$ and $f(x) = x - 1$ then $g(f(x)) = x^2 - 1$.

ANSWER

1. T 2. T 3. F 4. F 5. T 6. F

QUESTION

State in words the converse of each statement and say if true or false. If the converse is false, give a reason.

1. If a number ends in zero then it is divisible by 5.
2. If n is a prime number greater than 2 then n is an odd number.
3. $x = 3 \Rightarrow x^2 = 9$
4. If a and b are odd numbers then $a + b$ is even.
5. If 3 is a root of $x^2 + x - k = 0$ then k is a multiple of 3.

ANSWER

1. If a number is divisible by 5, then it ends in 0. False. e.g. 15
2. If n is an odd number greater than 2, then it is prime. False. e.g. 15
3. $x^2 = 9 \Rightarrow x = 3$. False. e.g. $x = -3$
4. If $a + b$ is even then a and b are odd. False. e.g. 2 and 4
5. If k is a multiple of 3 then 3 is a root of $x^2 + x - k = 0$. False e.g. $x^2 + x - 3 = 0$

Proof, Logic & Number Theory
Logic & Proof

QUESTION

In this exercise, state whether the implication can be replaced by the two way implication.

1. $a = b \Rightarrow a + c = b + c.$
2. $x = y \Rightarrow -x = -y.$
3. $n = -3 \Rightarrow n^2 = 9.$
4. If n is odd then n^2 is odd, ($n \in \mathbb{Z}$).
5. If a triangle ABC is right-angled at A $\Rightarrow a^2 = b^2 + c^2$.
6. If $y = 1 - x^2$ then $\frac{dy}{dx} = -2x$.

ANSWER

1. Yes 2. Yes 3. No 4 Yes 5. Yes 6. No.

QUESTION

USE CONTRAPOSITIVE TO PROVE THAT

1. $\sqrt{3}$ is irrational.
2. If n^3 is odd where n is an integer then n is odd.
3. If $x, y \in \mathbb{R}$ such that $x + y$ is irrational, then at least one of x and y is irrational.

ANSWER

1. Proof 2. Proof 3. Proof

Proof, Logic & Number Theory
Logic & Proof

QUESTION

1. Which of the following is a negation of "All boys are adventurous" ?
   (a) No boys are adventurous.
   (b) All boys are unadventurous.
   (c) Some boys are not adventurous.
   (d) No boys are unadventurous.

2. Which of these is a negation of "No visitors may walk on the grass" ?
   (a) All visitors may walk on the grass.
   (b) Some visitors may not walk on the grass.
   (c) All visitors may not walk on the grass.
   (d) Some visitors may walk on the grass.

3. Write down the negation of each of the following statements:-
   (a) For all real x, x^2 is positive.
   (b) Some pupils find mathematics difficult.
   (c) No dogs like cats.
   (d) There exists a positive integer x such that $x + 3 > 0$.
   (e) Every parallelogram has half turn symmetry.
   (f) No schoolboy lies.
   (g) A number which has zero in the units place is divisible by five.
   (h) All numbers of the form $2^n - 1$, (n an integer), are prime.

ANSWERS

1. (c) 2. (d)

3. (a) For some real x, x^2 is not positive.
   (b) No pupils find mathematics difficult.
   (c) Some dogs like cats.
   (d) There is no positive integer x such that $x + 3 > 0$.
   (e) Some parallelograms do not have half turn symmetry.
   (f) Some school boys lie.
   (g) Some numbers withh a zero in the units place are not divisible by 5.
   (h) Some numbers of the form $2^n - 1$, (n an integer), are not prime.

QUESTION

State the **converse** of each of the following, and show by a counter example that the converse is false.

1. (a) If a number ends in 0, it is divisible by 5.
   (b) All primes greater than 2 are odd numbers.
   (c) If a quadrilateral is a square, its diagonals intersect at right angles.
   (d) $x = 3 \Rightarrow x^2 = 9$.
   (e) If two numbers are odd then their sum is even.
   (f) If two integers are even, then their product is even.

Proof, Logic & Number Theory
Logic & Proof

ANSWER 1

1. (a) If a number is divisible by 5, it ends in zero. (e.g. 15)
   (b) If a number is odd, it is aprime number greater than 2. (e.g. 21)
   (c) If the diagonals of a quadrilateral intersect at right angles, the quadrilateral is a square. (e.g. a rhombus)
   (d) If $x^2 = 9$, $x = 3$ (e.g. $x = 3$)
   (e) If the sum of two numbers is even, => the numbers are odd. (6 & 4)
   (f) If the product of two numbers is even, the numbers are even. (6 & 5)

QUESTION 1

1. For each of these, say whether the first statement is:-
   (i) a necessary condition, (ii) a sufficient condition,
   (iii) both necessary and sufficient, (iv) neither,
   for the second condition condition.
   (a) p: there are more than 8 people in this room
   q: there are 9 people in this room.
   (b) p: ABCD is a parallelogram
   q: the diagonals of ABCD are perpendicular.

2. Which of the following statements are necessary or/and sufficient for the statement q: "natural number n is divisible by 6" to be correct ?
   (a) p: n is divisible by 3
   (b) p: n is divisible by 9
   (c) p: n is divisible by 12
   (d) p: n^2 is divisible by 12
   (e) p: $n = 384$
   (f) p: n is even and divisible by 3
   (g) $n = m(m + 1)(m + 2)$, where m is some natural number.

ANSWER 1

1. (a) necessary
   (b) neither

2. (a) necessary (b) none of these (c) sufficient
   (d) necessary and sufficient (e) sufficient
   (f) necessary and sufficient (g) neither (h) sufficient

Proof, Logic & Number Theory
Logic & Proof

QUESTION

1. Prove, using the contrapositive,
   - (a) that if x and y are integers and xy is odd, then both x and y are odd.
   - (b) that every prime number greater than 3 is of the form $6n \pm 1$, where n is a positive integer.
   - (c) that if n is a natural number such that n^2 is even, then n is even.

2. Prove, by ~~contradiction,~~ contrapositive
   - (a) that if x and y are integers such that $x + y$ is odd, then one of them must be odd and one must be even.
   - (b) that if x and y are real numbers such that $x + y$ is irrational, then at least one of x, y is irrational.
   - (c) that if m and n are integers such that mn^2 is even, then at least one of m or n is even.
   - (d) that if $\sin\theta \neq 0$, then $\theta \neq k\pi$ for any integer k.

ANSWER

ALL PROOFS

Proof, Logic & Number Theory
Euclidean Algorithm

QUESTION

Use the division identity for the following

1. $a = 75$ and $b = 12$
2. $a = 327$ and $b = 13$
3. $a = 392$ and $b = 19$

ANSWER

1. $75 = 6.12 + 3$ 2. $327 = 25.13 + 2$ 3. $392 = 20.19 + 12$

QUESTION

1. Find the G.C.D. of

   (i) (15, 27) (ii) (16, 42) (iii) (72, 108)

2. Use the Euclidean Algorithm to find the G.C.D. of

   (i) (1219, 901) (ii) (4277, 2821) (iii) (5213, 2867)

ANSWER

1. (i) 3 (ii) 2 (iii) 36 2. (i) 53 (ii) 91 (iii) 1

QUESTION

1. Express the G.C.D.'s found in [redacted] the 1st exercise Question 2 as a linear combination of the original numbers.

2. (a) Use the Euclidean Algorithm to find the G.C.D.of (7293, 798).

   (b) Hence find the integers x and y to write this G.C.D. in the form $x.7293 + y.798$.

Proof, Logic & Number Theory
Euclidean Algorithm

ANSWERS

1. (a) $53 = 3 \times 1219 - 4 \times 901$ (b) $91 = 2 \times 4277 - 3 \times 2821$
   (c) $1 = -952 \times 5213 + 1731 \times 2867$

2. (a) 3 (b) $x = -115, y = 1051$

Proof, Logic & Number Theory
Number Bases

Questions

1. Each of the following numbers are in base 10. Convert them to the base given in brackets next to them:

   (a) 81 (2)   (b) 579 (5)   (c) 1064 (7)   (d) 15287 (9)

2. Express the following numbers in base 10:

   (a) 1234_7 (b) 777_8 (c) 110110_2
   (d) $t81e_{12}$ where t and e are digits representing 10 and 11 respectively.

3. Change the following numbers to the base indicated:

   (a) 63_{10} to base 2   (b) 333_{10} to base 4   (c) 1727_{10} to base 12
   (d) 626_7 to base 5   (e) 401_6 to base 7
   (f) $tt5_{12}$ to base 6 where t is a digit representing 10.

Answers

1(a) 1010001_2 (b) 4304_5 (c) 3050_7 (d) 22865_9

2(a) 466_{10} (b) 511_{10} (c) 54_{10} (d) 18455_{10}

3(a) 111111_2 (b) 11031_4 (c) eee_{12} (e is a digit representing 11)
(d) 2224_5 (e) 265_7 (f) 11125_6