ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Systems of Linear Equations

1. $x + y + z = 10$
$2x - y + 3z = 4$
$x + 2z = 20$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 4 \\ 1 & 0 & 2 & 20 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -16 \\ 0 & -1 & 1 & 10 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to 3R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -16 \\ 0 & 0 & 2 & 46 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad 2z = 46 \quad \Rightarrow \quad z = 23$

$R_2 \quad \Rightarrow \quad -3y + z = -16 \quad \Rightarrow \quad -3y + 23 = -16 \quad \Rightarrow \quad -3y = -39 \quad \Rightarrow \quad y = 13$

$R_1 \quad \Rightarrow \quad x + y + z = 10 \quad \Rightarrow \quad x + 12 + 23 = 10 \quad \Rightarrow \quad x + 36 = 10 \quad \Rightarrow \quad x = -26$

Hence $x = -26$, $y = 13$ and $z = 23$.

Note

The solutions can be checked by substituting the values of *x*, *y* and *z* into the original three equations.

2. $x + y + 3z = 2$
$2x + y + z = 2$
$3x + 2y + 5z = 5$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 5 & 5 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & -5 & -2 \\ 0 & -1 & -4 & -1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & -5 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad z = 1$

$R_2 \quad \Rightarrow \quad -y - 5z = -2 \quad \Rightarrow \quad -y - 5(1) = -2 \quad \Rightarrow \quad -y - 5 = -2 \quad \Rightarrow \quad -y = 3$
$\Rightarrow \quad y = -3$

$R_1 \quad \Rightarrow \quad x + y + 3z = 2 \quad \Rightarrow \quad x + (-3) + 3(1) = 2 \quad \Rightarrow \quad x - 3 + 3 = 2 \quad \Rightarrow \quad x = 2$

Hence $x = 2$, $y = -3$ and $z = 1$.

Note

The solutions can be checked by substituting the values of *x*, *y* and *z* into the original three equations.

3. $x + y = 4$
$x + 2y - z = 5$
$3y - 2z = 6.$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 1 & 2 & -1 & 5 \\ 0 & 3 & -2 & 6 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -2 & 6 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad z = 3$

$R_2 \quad \Rightarrow \quad y - z = 1 \quad \Rightarrow \quad y - 3 = 1 \quad \Rightarrow \quad y = 4$

$R_1 \quad \Rightarrow \quad x + y = 4 \quad \Rightarrow \quad x + 4 = 4 \quad \Rightarrow \quad x = 0$

Hence $x = 0$, $y = 4$ and $z = 3$.

<u>Note</u>

The solutions can be checked by substituting the values of *x*, *y* and *z* into the original three equations.

4.

$$\begin{aligned} x \qquad - \; z &= 2 \\ 2y - 3z &= 6 \\ 2x + y + z &= 1 \end{aligned}$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 2 & -3 & 6 \\ 2 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 2R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 1 & 3 & -3 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to 2R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 9 & -12 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad 9z = -12 \quad \Rightarrow \quad z = -\dfrac{4}{3}$

$R_2 \quad \Rightarrow \quad 2y - 3z = 6 \quad \Rightarrow \quad 2y - 3\left(-\dfrac{4}{3}\right) = 6 \quad \Rightarrow \quad 2y + 4 = 6 \quad \Rightarrow \quad 2y = 2 \quad \Rightarrow \quad y = 1$

$R_1 \quad \Rightarrow \quad x - z = 2 \quad \Rightarrow \quad x - \left(-\dfrac{4}{3}\right) = 2 \quad \Rightarrow \quad x + \dfrac{4}{3} = 2 \quad \Rightarrow \quad x = \dfrac{2}{3}$

Hence $x = \dfrac{2}{3}$, $y = 1$ and $z = -\dfrac{4}{3}$.

<u>Note</u>

The solutions can be checked by substituting the values of *x*, *y* and *z* into the original three equations.

5. $x + y + z = 0$

$2x - y + z = -1\cdot 1$

$x + 3y + 2z = 0\cdot 9$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1\cdot 1 \\ 1 & 3 & 2 & 0\cdot 9 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1\cdot 1 \\ 0 & 2 & 1 & 0\cdot 9 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to 3R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1\cdot 1 \\ 0 & 0 & 1 & 0\cdot 5 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad z = 0\cdot 5$

$R_2 \quad \Rightarrow \quad -3y - z = -1\cdot 1 \quad \Rightarrow \quad -3y - 0\cdot 5 = -1\cdot 1 \quad \Rightarrow \quad -3y = -0\cdot 6 \quad \Rightarrow \quad y = 0\cdot 2$

$R_1 \quad \Rightarrow \quad x + y + z = 0 \quad \Rightarrow \quad x + 0\cdot 2 + 0\cdot 5 = 0 \quad \Rightarrow \quad x + 0\cdot 7 = 0 \quad \Rightarrow \quad x = -0\cdot 7$

Hence $x = -0\cdot 7$, $y = 0\cdot 2$ and $z = 0\cdot 5$.

<u>Note</u>

The solutions can be checked by substituting the values of *x*, *y* and *z* into the original three equations.

6. $x + y - z = 6$

$2x - 3y + 2z = 2$

$-5x + 2y - 4z = 1$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & 2 & 2 \\ -5 & 2 & -4 & 1 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 5R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 7 & -9 & 31 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \to 5R_3 + 7R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 0 & -17 & 85 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad -17z = 85 \quad \Rightarrow \quad z = -5$

$R_2 \quad \Rightarrow \quad -5y + 4z = -10 \quad \Rightarrow \quad -5y + 4(-5) = -10 \quad \Rightarrow \quad -5y - 20 = -10$

$\Rightarrow \quad -5y = 10$

$\Rightarrow \quad y = -2$

$R_1 \quad \Rightarrow \quad x + y - z = 6 \quad \Rightarrow \quad x + (-2) - (-5) = 6 \quad \Rightarrow \quad x - 2 + 5 = 6 \quad \Rightarrow \quad x + 3 = 6$

$\Rightarrow \quad x = 3$

Hence $x = 3$, $y = -2$ and $z = -5$.

Note

The solutions can be checked by substituting the values of *x*, *y* and *z* into the original three equations.

7.
$$\begin{aligned} x+y-z &= 3 \\ y+z &= -2 \\ x-3z &= 7 \end{aligned}$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & -3 & 7 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & -2 & 4 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad -z = 2 \quad \Rightarrow \quad z = -2$

$R_2 \quad \Rightarrow \quad y+z=-2 \quad \Rightarrow \quad y+(-2)=-2 \quad \Rightarrow \quad y-2=-2 \quad \Rightarrow \quad y=0$

$R_1 \quad \Rightarrow \quad x+y-z=3 \quad \Rightarrow \quad x+0-(-2)=3 \quad \Rightarrow \quad x+2=3 \quad \Rightarrow \quad x=1$

Hence $x=1$, $y=0$ and $z=-2$.

<u>Note</u>

The solutions can be checked by substituting the values of *x*, *y* and *z* into the original three equations.

8. $x + y = 1$

$2x + 3y + z = 2$

$2x + 2y + z = 1$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad z = -1$

$R_2 \quad \Rightarrow \quad y + z = 0 \quad \Rightarrow \quad y + (-1) = 0 \quad \Rightarrow \quad y - 1 = 0 \quad \Rightarrow \quad y = 1$

$R_1 \quad \Rightarrow \quad x + y = 1 \quad \Rightarrow \quad x + 1 = 1 \quad \Rightarrow \quad x = 0$

Hence $x = 0$, $y = 1$ and $z = -1$.

<u>Notes</u>

(1) The matrix reduces to upper triangular form after the first set of row operations.

(2) The solutions can be checked by substituting the values of *x*, *y* and *z* into the original three equations.

9. $x + 2y + 3z = 3$

$2x - y + 4z = 5$

$x - 3y + 2\lambda z = 2$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & -1 & 4 & 5 \\ 1 & -3 & 2\lambda & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -5 & -2 & -1 \\ 0 & -5 & 2\lambda - 3 & -1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -5 & -2 & -1 \\ 0 & 0 & 2\lambda - 1 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Redundancy will occur when all the entries in R_3 are zero.

Hence redundancy will occur when $2\lambda - 1 = 0 \quad \Rightarrow \quad 2\lambda = 1 \quad \Rightarrow \quad \lambda = \frac{1}{2}$

10.(a) (i)

$$x + 2y - z = -3$$
$$4x - 2y + 3z = 11$$
$$3x + y + 2\lambda z = 8$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 4 & -2 & 3 & 11 \\ 3 & 1 & 2\lambda & 8 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -10 & 7 & 23 \\ 0 & -5 & 2\lambda + 3 & 17 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to 2R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -10 & 7 & 23 \\ 0 & 0 & 4\lambda - 1 & 11 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \qquad ...(*)$$

$R_3 \quad \Rightarrow \quad (4\lambda - 1)z = 11 \quad \Rightarrow \quad z = \dfrac{11}{4\lambda - 1}$

(ii) The system of equations is inconsistent when $4\lambda - 1 = 0 \quad \Rightarrow \quad 4\lambda = 1 \quad \Rightarrow \quad \lambda = \dfrac{1}{4}$

(b) When $\lambda = -2 \cdot 5$, the matrix at (*) is $\left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -10 & 7 & 23 \\ 0 & 0 & -11 & 11 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$R_3 \quad \Rightarrow \quad -11z = 11 \quad \Rightarrow \quad z = -1$

$R_2 \quad \Rightarrow \quad -10y + 7z = 23 \quad \Rightarrow \quad -10y + 7(-1) = 23 \quad \Rightarrow \quad -10y - 7 = 23 \quad \Rightarrow \quad -10y = 30$
$\Rightarrow \quad y = -3$

$R_1 \quad \Rightarrow \quad x + 2y - z = -3 \quad \Rightarrow \quad x + 2(-3) - (-1) = -3 \quad \Rightarrow \quad x - 6 + 1 = -3 \quad \Rightarrow \quad x - 5 = -3$
$\Rightarrow \quad x = 2$

Hence $x = 2$, $y = -3$ and $z = -1$.

11. $x - 2y + z = -4$

$3x - 5y - 2z = 1$

$-7x + 11y + \lambda z = -11$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 3 & -5 & -2 & 1 \\ -7 & 11 & \lambda & -11 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 7R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -5 & 13 \\ 0 & -3 & \lambda + 7 & -39 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -5 & 13 \\ 0 & 0 & \lambda - 8 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Redundancy will occur when all the entries in R_3 are zero.

Hence redundancy will occur when $\lambda - 8 = 0 \quad \Rightarrow \quad \lambda = 8$

12.(a) (i)

$$x + y + z = 2$$
$$4x + 3y - \lambda z = 4$$
$$5x + 6y + 8z = 11$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 3 & -\lambda & 4 \\ 5 & 6 & 8 & 11 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -\lambda - 4 & -4 \\ 0 & 1 & 3 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -\lambda - 4 & -4 \\ 0 & 0 & -\lambda - 1 & -3 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \qquad ...(*)$$

$$R_3 \quad \Rightarrow \quad (-\lambda - 1)z = -3 \quad \Rightarrow \quad z = \frac{-3}{-\lambda - 1} = \frac{3}{\lambda + 1}$$

(ii) The system of equations is inconsistent when $\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -1$

(b) When $\lambda = 2$, the matrix at (*) is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -6 & -4 \\ 0 & 0 & -3 & -3 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$$R_3 \quad \Rightarrow \quad -3z = -3 \quad \Rightarrow \quad z = 1$$

$$R_2 \quad \Rightarrow \quad -y - 6z = -4 \quad \Rightarrow \quad -y - 6(1) = -4 \quad \Rightarrow \quad -y - 6 = -4 \quad \Rightarrow \quad -y = 2$$
$$\Rightarrow \quad y = -2$$

$$R_1 \quad \Rightarrow \quad x + y + z = 2 \quad \Rightarrow \quad x + (-2) + 1 = 2 \quad \Rightarrow \quad x - 2 + 1 = 2 \quad \Rightarrow \quad x - 1 = 2$$
$$\Rightarrow \quad x = 3$$

Hence $x = 3$, $y = -2$ and $z = 1$.

13.(a) $x - y + z = 1$

$x + y + 2z = 0$

$2x - y + az = 2$

When $a = 3$, the matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & 3 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to 2R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \Rightarrow z = 1$

$R_2 \Rightarrow 2y + z = -1 \Rightarrow 2y + 1 = -1 \Rightarrow 2y = -2 \Rightarrow y = -1$

$R_1 \Rightarrow x - y + z = 1 \Rightarrow x - (-1) + 1 = 1 \Rightarrow x + 1 + 1 = 1 \Rightarrow x + 2 = 1$

$\Rightarrow x = -1$

Hence $x = -1$, $y = -1$ and $z = 1$.

(b) When $a = 2 \cdot 5$, the matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & 2 \cdot 5 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & 0 \cdot 5 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to 2R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \Rightarrow 0 = 1$ hence the system of equations is inconsistent when $a = 2 \cdot 5$ and has no solution.

14.(a) $x + y + 3z = 1$

$3x + ay + z = 1$

$x + y + z = -1$

When $a = 6$, the matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 3 & 6 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 3 & -8 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Note that the matrix is already in upper triangular form.

$R_3 \Rightarrow -2z = -2 \Rightarrow z = 1$

$R_2 \Rightarrow 3y - 8z = -2 \Rightarrow 3y - 8(1) = -2 \Rightarrow 3y - 8 = -2 \Rightarrow 3y = 6$
$\Rightarrow y = 2$

$R_1 \Rightarrow x + y + 3z = 1 \Rightarrow x + 2 + 3(1) = 1 \Rightarrow x + 2 + 3 = 1 \Rightarrow x + 5 = 1$
$\Rightarrow x = -4$

Hence $x = -4$, $y = 2$ and $z = 1$.

(b) When $a = 3$, the matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 3 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 0 & -8 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \Rightarrow -2z = -2 \Rightarrow z = 1$

$R_2 \Rightarrow -8z = -2 \Rightarrow z = \frac{1}{4}$ which contradicts the value of z found from R_3.

Hence the system of equations is inconsistent when $a = 3$ and has no solution.

Note

An alternative way of determining that the system of equations is inconsistent is as follows.

When $a = 3$, the matrix of coefficients is
$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 3 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 0 & -8 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Performing the row operation $R_3 \to 4R_3 - R_2$ leads to the matrix $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 0 & -8 & -2 \\ 0 & 0 & 0 & -6 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$R_3 \quad \Rightarrow \quad 0 = -6$ hence the system of equations is inconsistent when $a = 3$ and has no solution.

15.(a) $x + y + 2z = 1$

$2x + \lambda y + z = 0$

$3x + 3y + 9z = 5$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & \lambda & 1 & 0 \\ 3 & 3 & 9 & 5 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \lambda - 2 & -3 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \qquad ...(*)$$

Note that the matrix is already in upper triangular form.

$R_3 \quad \Rightarrow \quad 3z = 2 \quad \Rightarrow \quad z = \frac{2}{3}$

$R_2 \quad \Rightarrow \quad (\lambda - 2)y - 3z = -2 \quad \Rightarrow \quad (\lambda - 2)y - 3\left(\frac{2}{3}\right) = -2$

$\Rightarrow \quad (\lambda - 2)y - 2 = -2$

$\Rightarrow \quad (\lambda - 2)y = 0$

$\Rightarrow \quad y = 0 \qquad$ [since $\lambda \neq 2$ and so $\lambda - 2 \neq 0$]

$R_1 \quad \Rightarrow \quad x + y + 2z = 1 \quad \Rightarrow \quad x + 0 + 2\left(\frac{2}{3}\right) = 1 \quad \Rightarrow \quad x + \frac{4}{3} = 1 \quad \Rightarrow \quad x = -\frac{1}{3}$

Hence $x = -\frac{1}{3}$, $y = 0$ and $z = \frac{2}{3}$.

(b) When $\lambda = 2$, the matrix at (*) is $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

Performing the row operation $R_3 \to R_3 + R_2$ leads to the matrix $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

All the entries in R_3 are zero, so the system contains a redundant equation when $\lambda = 2$ and has an infinite number of solutions.

16.(a) $2x - y + 2z = 1$
$x + y - 2z = 2$
$x - 2y + 4z = -1$

The matrix of coefficients is
$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & -2 & 4 & -1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \rightarrow 2R_2 - R_1 \\ R_3 \rightarrow 2R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & -3 & 6 & -3 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \rightarrow R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

All the entries in R_3 are zero, so the system contains a redundant equation and has an infinite number of possible solutions.

(b) Let $z = t$.

$R_2 \Rightarrow 3y - 6z = 3 \Rightarrow 3y - 6t = 3 \Rightarrow 3y = 6t + 3 \Rightarrow y = 2t + 1$

$$\begin{aligned} R_1 \Rightarrow 2x - y + 2z = 1 &\Rightarrow 2x - (2t+1) + 2t = 1 \\ &\Rightarrow 2x - 2t - 1 + 2t = 1 \\ &\Rightarrow 2x - 1 = 1 \\ &\Rightarrow 2x = 2 \\ &\Rightarrow x = 1 \end{aligned}$$

Hence $x = 1$ and $y = 2t + 1$.

17.(a) $4x + 6z = 1$

$2x - 2y + 4z = -1$

$-x + y + \lambda z = 2$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 2 & -2 & 4 & -1 \\ -1 & 1 & \lambda & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to 2R_2 - R_1 \\ R_3 \to 4R_3 + R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 0 & -4 & 2 & -3 \\ 0 & 4 & 4\lambda+6 & 9 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 0 & -4 & 2 & -3 \\ 0 & 0 & 4\lambda+8 & 6 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \Rightarrow (4\lambda + 8)z = 6 \Rightarrow z = \frac{6}{4\lambda+8} = \frac{3}{2\lambda+4}$

$R_2 \Rightarrow -4y + 2z = -3 \Rightarrow -4y + 2\left(\frac{3}{2\lambda+4}\right) = -3$

$\Rightarrow -4y + \frac{6}{2\lambda+4} = -3$ [multiply all terms by $(2\lambda+4)$]

$\Rightarrow -4(2\lambda+4)y + 6 = -3(2\lambda+4)$

$\Rightarrow -4(2\lambda+4)y = -3(2\lambda+4) - 6$

$\Rightarrow -4(2\lambda+4)y = -6\lambda - 18$

$\Rightarrow y = \frac{-6\lambda-18}{-4(2\lambda+4)} = \frac{-6\lambda-18}{-8\lambda-16} = \frac{6\lambda+18}{8\lambda+16} = \frac{3\lambda+9}{4\lambda+8}$

$R_1 \Rightarrow 4x + 6z = 1 \Rightarrow 4x + 6\left(\frac{3}{2\lambda+4}\right) = 1$

$\Rightarrow 4x + \frac{18}{2\lambda+4} = 1$ [multiply all terms by $(2\lambda+4)$]

$\Rightarrow 4(2\lambda+4)x + 18 = 2\lambda + 4$

$\Rightarrow 4(2\lambda+4)x = 2\lambda + 4 - 18$

$\Rightarrow 4(2\lambda+4)x = 2\lambda - 14$

$\Rightarrow x = \frac{2\lambda-14}{4(2\lambda+4)} = \frac{2\lambda-14}{8\lambda+16} = \frac{\lambda-7}{4\lambda+8}$

Hence $x = \frac{\lambda-7}{4\lambda+8}$, $y = \frac{3\lambda+9}{4\lambda+8}$ and $z = \frac{3}{2\lambda+4}$.

(b) From the expression $z = \dfrac{3}{2\lambda + 4}$, the system of equations is inconsistent when

$$2\lambda + 4 = 0 \quad \Rightarrow \quad 2\lambda = -4 \quad \Rightarrow \quad \lambda = -2$$

The system of equations is inconsistent when $\lambda = -2$ and has no solution.

(c) When $\lambda = -2 \cdot 1$: $x = \dfrac{\lambda - 7}{4\lambda + 8} = \dfrac{-2 \cdot 1 - 7}{4(-2 \cdot 1) + 8} = \dfrac{-9 \cdot 1}{-0 \cdot 4} = 22 \cdot 75$

$$y = \frac{3\lambda + 9}{4\lambda + 8} = \frac{3(-2 \cdot 1) + 9}{4(-2 \cdot 1) + 8} = \frac{2 \cdot 7}{-0 \cdot 4} = -6 \cdot 75$$

$$z = \frac{3}{2\lambda + 4} = \frac{3}{2(-2 \cdot 1) + 4} = \frac{3}{-0 \cdot 2} = -15$$

(d) When $\lambda = -1 \cdot 9$ the solution is $x = -22 \cdot 25$, $y = 8 \cdot 25$, $z = 15$.
When $\lambda = -2 \cdot 1$ the solution is $x = 22 \cdot 75$, $y = -6 \cdot 75$, $z = -15$.

Comparing the two sets of solutions for x, y and z, we can see that a small change in the value of the parameter λ (from $\lambda = -1 \cdot 9$ to $\lambda = -2 \cdot 1$) produces a relatively large change in the values of x, y and z.

Note

We say that the system of equations **is ill-conditioned** near $\lambda = -2$.