ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Sequences and Series

1. We have an arithmetic sequence with $a = 2$ and $u_{20} = 97$.

Make up an equation for u_{20} using the formula $u_n = a + (n-1)d$.

$$u_{20} = 97 \quad \Rightarrow \quad a + 19d = 97 \quad \Rightarrow \quad 2 + 19d = 97 \quad \Rightarrow \quad 19d = 95 \quad \Rightarrow \quad d = 5$$

We can now find S_{50} using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$.

$$S_{50} = \frac{50}{2}[2(2) + 49(5)] = 25(249) = 6225$$

2. We have an arithmetic series with $S_{20} = 320$ and $u_{21} = 37$.

Make up an equation for S_{20} using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$.

$$S_{20} = 320 \quad \Rightarrow \quad \frac{20}{2}(2a + 19d) = 320 \quad \Rightarrow \quad 10(2a + 19d) = 320$$

$$\Rightarrow \quad 2a + 19d = 32 \qquad ...(1)$$

Make up an equation for u_{21} using the formula $u_n = a + (n-1)d$.

$$u_{21} = 37 \quad \Rightarrow \quad a + 20d = 37 \qquad ...(2)$$

Solving equations (1) and (2) gives $a = -3$ and $d = 2$.

We can now find S_{10} using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$.

$$S_{10} = \frac{10}{2}[2(-3) + 9(2)] = 5(12) = 60$$

3.(a) We have an arithmetic sequence with $u_1 = 45$ and $u_3 = 5$.

$u_1 = 45 \quad \Rightarrow \quad a = 45$

Make up an equation for u_3 using the formula $u_n = a + (n-1)d$.

$u_3 = 5 \quad \Rightarrow \quad a + 2d = 5 \quad \Rightarrow \quad 45 + 2d = 5 \quad \Rightarrow \quad 2d = -40 \quad \Rightarrow \quad d = -20$

We can now find u_{11} using the formula $u_n = a + (n-1)d$.

$u_{11} = a + 10d = 45 + 10(-20) = -155$

(b) We have a geometric sequence of positive numbers with $v_1 = 45$ and $v_3 = 5$.

$v_1 = 45 \quad \Rightarrow \quad a = 45$

Make up an equation for v_3 using the formula $v_n = ar^{n-1}$.

$v_3 = 5 \quad \Rightarrow \quad ar^2 = 5 \quad \Rightarrow \quad 45r^2 = 5 \quad \Rightarrow \quad r^2 = \frac{1}{9} \quad \Rightarrow \quad r = \pm\frac{1}{3}$

Hence $r = \frac{1}{3}$ since all terms are positive meaning that $r > 0$.

Note that $-1 < r < 1$, so a sum to infinity exists.

$$S_\infty = \frac{a}{1-r} = \frac{45}{1-{}^1\!/_3} = \frac{45}{{}^2\!/_3} = \frac{135}{2} \quad or \quad 67\frac{1}{2}$$

4. We have a geometric sequence with $u_2 = -6$ and $u_3 = 3$.

Make up equations for u_2 and u_3 using the formula $u_n = ar^{n-1}$.

$u_2 = -6 \Rightarrow ar = -6$ ...(1)

$u_3 = 3 \Rightarrow ar^2 = 3$ ...(2)

$(2) \div (1) \Rightarrow \dfrac{ar^2}{ar} = \dfrac{3}{-6} \Rightarrow r = -\dfrac{1}{2}$

$-1 < r < 1$, so a sum to infinity exists.

To find a, substitute $r = -\dfrac{1}{2}$ into (1) $\Rightarrow a\left(-\dfrac{1}{2}\right) = -6 \Rightarrow a = 12$

$$S_\infty = \frac{a}{1-r} = \frac{12}{1-\left(-\frac{1}{2}\right)} = \frac{12}{3/2} = 8$$

5.(a) We have a geometric sequence with $u_2 = 108$ and $u_5 = 4$.

Make up equations for u_2 and u_5 using the formula $u_n = ar^{n-1}$.

$u_2 = 108 \Rightarrow ar = 108$ ...(1)

$u_5 = 4 \Rightarrow ar^4 = 4$ ...(2)

$(2) \div (1) \Rightarrow \dfrac{ar^4}{ar} = \dfrac{4}{108} \Rightarrow r^3 = \dfrac{1}{27} \Rightarrow r = \dfrac{1}{3}$

(b) $-1 < r < 1$, so a sum to infinity exists.

(c) To find a, substitute $r = \dfrac{1}{3}$ into (1) $\Rightarrow a\left(\dfrac{1}{3}\right) = 108 \Rightarrow a = 324$

$$S_\infty = \frac{a}{1-r} = \frac{324}{1-1/3} = \frac{324}{2/3} = 486$$

6.(a) We have an arithmetic sequence with $u_5 = -6$ and $u_{12} = -34$.

Make up equations for u_5 and u_{12} using the formula $u_n = a + (n-1)d$.

$u_5 = -6 \quad \Rightarrow \quad a + 4d = -6 \qquad ...(1)$

$u_{12} = -34 \quad \Rightarrow \quad a + 11d = -34 \qquad ...(2)$

Solving equations (1) and (2) gives $a = 10$ and $d = -4$.

The first term is 10 and the common difference is −4.

(b) Make up an equation for S_n using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$.

$$S_n = -144 \quad \Rightarrow \quad \frac{n}{2}[2(10) + (n-1)(-4)] = -144$$

$$\Rightarrow \quad \frac{n}{2}[20 - 4(n-1)] = -144$$

$$\Rightarrow \quad \frac{n}{2}(20 - 4n + 4) = -144$$

$$\Rightarrow \quad \frac{n}{2}(24 - 4n) = -144 \qquad \text{[now multiply out the brackets]}$$

$$\Rightarrow \quad 12n - 2n^2 = -144 \qquad \text{[now make the RHS} = 0\text{]}$$

$$\Rightarrow \quad -2n^2 + 12n + 144 = 0 \qquad [\times(-1)]$$

$$\Rightarrow \quad 2n^2 - 12n - 144 = 0 \qquad [\div 2]$$

$$\Rightarrow \quad n^2 - 6n - 72 = 0$$

$$\Rightarrow \quad (n+6)(n-12) = 0$$

$$\Rightarrow \quad n = -6, \; n = 12$$

Hence $n = 12$ since $n > 0$.

Note

The quadratic equation $n^2 - 6n - 72 = 0$ (or equivalent) can also be solved using the quadratic formula.

7. $S_n = 8n - n^2, \quad n \geq 1$

First evaluate S_1, S_2 and S_3.

$S_1 = 8(1) - 1^2 = 7 \qquad S_2 = 8(2) - 2^2 = 12 \qquad S_3 = 8(3) - 3^2 = 15$

S_n is the **sum** of the first *n* terms of a sequence $u_1, u_2, u_3, \ldots$

$S_1 = u_1 = 7$

$S_2 = u_1 + u_2 = 12 \quad \Rightarrow \quad 7 + u_2 = 12 \quad \Rightarrow \quad u_2 = 5$

$S_3 = u_1 + u_2 + u_3 = 15 \quad \Rightarrow \quad 7 + 5 + u_3 = 15 \quad \Rightarrow \quad 12 + u_3 = 15 \quad \Rightarrow \quad u_3 = 3$

Hence $u_1 = 7$, $u_2 = 5$ and $u_3 = 3$.

The sequence is an **arithmetic** sequence with first term $a = 7$ and common difference $d = -2$.

$$u_n = a + (n-1)d \quad \Rightarrow \quad \begin{aligned} u_n &= 7 + (n-1)(-2) \\ &= 7 - 2(n-1) \\ &= 7 - 2n + 2 \\ &= 9 - 2n \end{aligned}$$

Hence $u_n = 9 - 2n$.

8.(a) We have a geometric sequence with $a = 80$ and $r = \frac{1}{3}$.

(i) We can find u_7 using the formula $u_n = ar^{n-1}$.

$$u_7 = ar^6 = 80\left(\frac{1}{3}\right)^6 = 80\left(\frac{1}{729}\right) = \frac{80}{729}$$

(ii) Note that $-1 < r < 1$, so a sum to infinity exists.

$$S_\infty = \frac{a}{1-r} = \frac{80}{1-\frac{1}{3}} = \frac{80}{\frac{2}{3}} = 120$$

(b) We have an arithmetic sequence with $a = 80$ and $S_5 = 240$.

(i) Make up an equation for S_5 using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$.

$$\begin{aligned} S_5 = 240 \quad &\Rightarrow \quad \frac{5}{2}[2(80) + 4d] = 240 \\ &\Rightarrow \quad \frac{5}{2}(160 + 4d) = 240 \qquad \text{[now multiply out the brackets]} \\ &\Rightarrow \quad 400 + 10d = 240 \\ &\Rightarrow \quad 10d = -160 \\ &\Rightarrow \quad d = -16 \end{aligned}$$

(ii) $u_n = a + (n-1)d \quad \Rightarrow \quad u_n = 80 + (n-1)(-16)$

$$\begin{aligned} &= 80 - 16(n-1) \\ &= 80 - 16n + 16 \\ &= 96 - 16n \end{aligned}$$

Hence $u_n = 96 - 16n$.

(c) Make up an equation for S_n using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$.

$$
\begin{aligned}
S_n = 144 \quad &\Rightarrow \quad \frac{n}{2}[2(80) + (n-1)(-16)] = 144 \\
&\Rightarrow \quad \frac{n}{2}[160 - 16(n-1)] = 144 \\
&\Rightarrow \quad \frac{n}{2}(160 - 16n + 16) = 144 \\
&\Rightarrow \quad \frac{n}{2}(176 - 16n) = 144 && \text{[now multiply out the brackets]} \\
&\Rightarrow \quad 88n - 8n^2 = 144 && \text{[now make the RHS} = 0\text{]} \\
&\Rightarrow \quad -8n^2 + 88n - 144 = 0 && [\times(-1)] \\
&\Rightarrow \quad 8n^2 - 88n + 144 = 0 && [\div 8] \\
&\Rightarrow \quad n^2 - 11n + 18 = 0 \\
&\Rightarrow \quad (n-2)(n-9) = 0 \\
&\Rightarrow \quad n = 2,\ n = 9
\end{aligned}
$$

9. We have a geometric series with $a = 2048$ and $u_4 = 256$.

Make up an equation for u_4 using the formula $u_n = ar^{n-1}$.

$$u_4 = 256 \quad \Rightarrow \quad ar^3 = 256 \quad \Rightarrow \quad 2048r^3 = 256 \quad \Rightarrow \quad r^3 = \frac{1}{8} \quad \Rightarrow \quad r = \frac{1}{2}$$

Make up an equation for S_n using the formula $S_n = \dfrac{a(1-r^n)}{1-r}$.

$$S_n = 4088 \quad \Rightarrow \quad \frac{2048\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 4088$$

$$\Rightarrow \quad \frac{2048\left(1-\frac{1}{2^n}\right)}{\frac{1}{2}} = 4088 \qquad \left[\text{since } \left(\frac{1}{2}\right)^n = \frac{1^n}{2^n} = \frac{1}{2^n}\right]$$

$$\Rightarrow \quad 4096\left(1-\frac{1}{2^n}\right) = 4088$$

$$\Rightarrow \quad 1-\frac{1}{2^n} = \frac{511}{512} \quad \Rightarrow \quad \frac{1}{2^n} = \frac{1}{512} \quad \Rightarrow \quad 2^n = 512 \quad \Rightarrow \quad n = 9$$

Note

The equation $2^n = 512$ can also be solved by taking logarithms:

$$2^n = 512 \quad \Rightarrow \quad \ln 2^n = \ln 512 \quad \Rightarrow \quad n\ln 2 = \ln 512 \quad \Rightarrow \quad n = \frac{\ln 512}{\ln 2} = 9$$

10. We have an arithmetic sequence with $u_1 = a$, $u_2 = \frac{1}{a}$ and $u_3 = 1$.

For any arithmetic sequence, the common difference, *d*, is given by subtracting consecutive terms, so $d = u_2 - u_1 = \frac{1}{a} - a$ and $d = u_3 - u_2 = 1 - \frac{1}{a}$.

Hence $\frac{1}{a} - a = 1 - \frac{1}{a}$ [multiply all terms by *a* to clear the fractions]

$\Rightarrow\ 1 - a^2 = a - 1$ [now make the RHS $= 0$]

$\Rightarrow\ -a^2 - a + 2 = 0$ [$\times(-1)$]

$\Rightarrow\ a^2 + a - 2 = 0$

$\Rightarrow\ (a+2)(a-1) = 0$

$\Rightarrow\ a = -2,\ a = 1$

Hence $a = -2$ since $a < 0$.

The common difference, *d*, can be found using either $d = \frac{1}{a} - a$ or $d = 1 - \frac{1}{a}$.

$$d = 1 - \frac{1}{a} = 1 - \frac{1}{(-2)} = 1 + \frac{1}{2} = \frac{3}{2}$$

We must find the smallest value of *n* for which $S_n > 1000$.

Consider solving the equation $S_n = 1000$ using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$.

$S_n = 1000 \quad \Rightarrow\ \frac{n}{2}\left[2(-2) + (n-1)\frac{3}{2}\right] = 1000$

$\Rightarrow\ \frac{n}{2}\left[-4 + \frac{3}{2}(n-1)\right] = 1000$

$\Rightarrow\ \frac{n}{2}\left(-4 + \frac{3}{2}n - \frac{3}{2}\right) = 1000$

$\Rightarrow\ \frac{n}{2}\left(\frac{3}{2}n - \frac{11}{2}\right) = 1000$ [now multiply out the brackets]

$\Rightarrow\ \frac{3}{4}n^2 - \frac{11}{4}n = 1000$ [multiply all terms by 4 to clear the fractions]

$\Rightarrow\ 3n^2 - 11n = 4000$ [now make the RHS $= 0$]

$\Rightarrow\ 3n^2 - 11n - 4000 = 0$

The quadratic equation $3n^2 - 11n - 4000 = 0$ must be solved using the **quadratic formula**.

$a = 3,\ b = -11,\ c = -4000 \quad \Rightarrow \quad b^2 - 4ac = (-11)^2 - 4(3)(-4000) = 48\ 121$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-11) \pm \sqrt{48121}}{2(3)} = \frac{11 \pm \sqrt{48121}}{6} = 38 \cdot 39\ldots \text{ or } -34 \cdot 72\ldots$$

Hence $n = 38 \cdot 39\ldots$ since $n > 0$.

Interpreting this solution in context, the smallest value of *n* for which $S_n > 1000$ is $n = 39$.

11. For the arithmetic sequence, $u_n = a + (n-1)d$.

Hence $u_3 = a + 2d$, $u_7 = a + 6d$ and $u_{16} = a + 15d$.

Let v_1, v_2 and v_3 denote the first three terms of the geometric sequence.

Then $v_1 = a + 2d$, $v_2 = a + 6d$ and $v_3 = a + 15d$.

For any geometric sequence, the common ratio, r, is given by dividing consecutive terms, so $r = \frac{v_2}{v_1} = \frac{a+6d}{a+2d}$ and $r = \frac{v_2}{v_1} = \frac{a+15d}{a+6d}$.

Hence $\frac{a+6d}{a+2d} = \frac{a+15d}{a+6d}$ [cross-multiply to clear the fractions]

$\Rightarrow (a+6d)(a+6d) = (a+15d)(a+2d)$

$\Rightarrow a^2 + 12ad + 36d^2 = a^2 + 17ad + 30d^2$ [subtract a^2 from each side]

$\Rightarrow 12ad + 36d^2 = 17ad + 30d^2$

$\Rightarrow 6d^2 - 5ad = 0$

$\Rightarrow d(6d - 5a) = 0$

$\Rightarrow d = 0$ or $6d - 5a = 0$

$\Rightarrow d = 0$ or $6d = 5a$

$\Rightarrow d = 0$ or $a = \frac{6}{5}d$

Hence $a = \frac{6}{5}d$ since $d \neq 0$.

The common ratio, r, can be found by substituting $a = \frac{6}{5}d$ into either $r = \frac{a+6d}{a+2d}$ or $r = \frac{a+15d}{a+6d}$.

$$r = \frac{a+6d}{a+2d} = \frac{\frac{6}{5}d + 6d}{\frac{6}{5}d + 2d} = \frac{6d + 30d}{6d + 10d} = \frac{36d}{16d} = \frac{36}{16} = \frac{9}{4}$$

12.(a) $S_n = \sum_{r=1}^{n}(11-2r) = \sum_{r=1}^{n}11 - \sum_{r=1}^{n}2r$

$$= 11n - 2\sum_{r=1}^{n}r$$

$$= 11n - 2\left(\frac{n(n+1)}{2}\right)$$

$$= 11n - n(n+1)$$

$$= 11n - n^2 - n$$

$$= 10n - n^2$$

Hence $S_n = 10n - n^2$.

(b) $S_n = 21 \Rightarrow 10n - n^2 = 21$ [now make the RHS $= 0$]

$\Rightarrow -n^2 + 10n - 21 = 0$ [$\times(-1)$]

$\Rightarrow n^2 - 10n + 21 = 0$

$\Rightarrow (n-3)(n-7) = 0$

$\Rightarrow n = 3,\ n = 7$

13.(a) $\sum_{r=1}^{n}(6r^2 - r) = \sum_{r=1}^{n}6r^2 - \sum_{r=1}^{n}r$

$= 6\sum_{r=1}^{n}r^2 - \sum_{r=1}^{n}r$

$= 6\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}$

$= n(n+1)(2n+1) - \frac{n(n+1)}{2}$

$= \frac{n(n+1)(2n+1)}{1} - \frac{n(n+1)}{2}$

$= \frac{2n(n+1)(2n+1)}{2} - \frac{n(n+1)}{2}$ [making a common denominator]

$= \frac{2n(n+1)(2n+1) - n(n+1)}{2}$

$= \frac{n(n+1)\big(2(2n+1) - 1\big)}{2}$ [removing a common factor of $n(n+1)$ in the numerator]

$= \frac{n(n+1)(4n+1)}{2}$

$= \frac{1}{2}n(n+1)(4n+1)$

(b) Note that $\sum_{r=5}^{10}(6r^2 - r)$ is a **partial sum**.

$\sum_{r=5}^{10}(6r^2 - r) = \sum_{r=1}^{10}(6r^2 - r) - \sum_{r=1}^{4}(6r^2 - r)$

$= \frac{1}{2}(10)(10+1)(4(10)+1) - \frac{1}{2}(4)(4+1)(4(4)+1)$ [using the result in (a)]

$= \frac{1}{2}(10)(11)(41) - \frac{1}{2}(4)(5)(17)$

$= 2255 - 170$

$= 2085$

Note

The method of **proof by induction** could instead be used to prove the result in (a).

14.(a) $\sum_{r=1}^{n}(4-6r)=\sum_{r=1}^{n}4-\sum_{r=1}^{n}6r$

$$=4n-6\sum_{r=1}^{n}r$$

$$=4n-6\left(\frac{n(n+1)}{2}\right)$$

$$=4n-3n(n+1)$$

$$=4n-3n^2-3n$$

$$=n-3n^2$$

(b) $\sum_{r=1}^{n}(4-6r)=n-3n^2 \quad\Rightarrow\quad \sum_{r=1}^{2q}(4-6r)=2q-3(2q)^2$ [replacing n with $2q$]

$$=2q-3(4q^2)$$

$$=2q-12q^2$$

(c) Note that $\sum_{r=q+1}^{2q}(4-6r)$ is a **partial sum**.

$$\sum_{r=q+1}^{2q}(4-6r)=\sum_{r=1}^{2q}(4-6r)-\sum_{r=1}^{q}(4-6r)$$

$=(2q-12q^2)-(q-3q^2)$ [using the results in (a) and (b)]

$$=2q-12q^2-q+3q^2$$

$$=q-9q^2$$

15.(a) $\sum_{r=1}^{N} r = 210 \quad \Rightarrow \quad \frac{N(N+1)}{2} = 210$ [using the formula $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$]

$$\Rightarrow \quad N(N+1) = 420$$

$$\Rightarrow \quad N^2 + N = 420$$

$$\Rightarrow \quad N^2 + N - 420 = 0$$

$$\Rightarrow \quad (N+21)(N-20) = 0$$

$$\Rightarrow \quad N = -21, \; N = 20$$

Hence $N = 20$ since $N > 0$.

<u>Note</u>

The quadratic equation $N^2 + N - 420 = 0$ can also be solved using the quadratic formula.

(b) $\sum_{r=1}^{N} r^2$ can be evaluated using the formula $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

$$\sum_{r=11}^{N} r^2 = \sum_{r=1}^{20} r^2 = \frac{20(20+1)(2(20)+1)}{6} = \frac{20(21)(41)}{6} = 2870$$

16.(a) $\sum_{r=1}^{n}(r^3-3r)=\sum_{r=1}^{n}r^3-\sum_{r=1}^{n}3r$

$$=\sum_{r=1}^{n}r^3-3\sum_{r=1}^{n}r$$

$$=\frac{n^2(n+1)^2}{4}-3\left(\frac{n(n+1)}{2}\right)$$

$$=\frac{n^2(n+1)^2}{4}-\frac{3n(n+1)}{2}$$

$$=\frac{n^2(n+1)^2}{4}-\frac{6n(n+1)}{4}$$ [making a common denominator]

$$=\frac{n^2(n+1)^2-6n(n+1)}{4}$$

$$=\frac{n(n+1)\big(n(n+1)-6\big)}{4}$$ [removing a common factor of $n(n+1)$ in the numerator]

$$=\frac{n(n+1)(n^2+n-6)}{4}$$

$$=\frac{n(n+1)(n-2)(n+3)}{4}$$

(b) Note that $\sum_{r=5}^{15}(r^3-3r)$ is a **partial sum**.

$$\sum_{r=5}^{15}(r^3-3r)=\sum_{r=1}^{15}(r^3-3r)-\sum_{r=1}^{4}(r^3-3r)$$

$$=\frac{15(15+1)(15-2)(15+3)}{4}-\frac{4(4+1)(4-2)(4+3)}{4}$$ [using the result in (a)]

$$=\frac{15(16)(13)(18)}{4}-\frac{4(5)(2)(7)}{4}$$

$$=14\,040-70$$

$$=13\,970$$

__**Note**__

The method of **proof by induction** could instead be used to prove the result in (a).

17. $\sum_{r=1}^{n}(2r^3+r^2-r)=\sum_{r=1}^{n}2r^3+\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r$

$$=2\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r$$

$$=2\left(\frac{n^2(n+1)^2}{4}\right)+\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}$$

$$=\frac{n^2(n+1)^2}{2}+\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}$$

$$=\frac{3n^2(n+1)^2}{6}+\frac{n(n+1)(2n+1)}{6}-\frac{3n(n+1)}{6}$$ [making a common denominator]

$$=\frac{3n^2(n+1)^2+n(n+1)(2n+1)-3n(n+1)}{6}$$

$$=\frac{n(n+1)\big(3n(n+1)+(2n+1)-3\big)}{6}$$ [removing a common factor of $n(n+1)$]

$$=\frac{n(n+1)(3n^2+3n+2n+1-3)}{6}$$

$$=\frac{n(n+1)(3n^2+5n-2)}{6}$$

$$=\frac{n(n+1)(3n-1)(n+2)}{6} \quad or \quad \frac{n(n+1)(n+2)(3n-1)}{6}$$

18.(a) $S_n = \sum_{r=1}^{n}\left(r^2 + \frac{1}{3}r\right) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n}\frac{1}{3}r$

$= \sum_{r=1}^{n} r^2 + \frac{1}{3}\sum_{r=1}^{n} r$

$= \frac{n(n+1)(2n+1)}{6} + \frac{1}{3}\left(\frac{n(n+1)}{2}\right)$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{6}$

$= \frac{n(n+1)(2n+1) + n(n+1)}{6}$

$= \frac{n(n+1)\big((2n+1)+1\big)}{6}$ [removing $n(n+1)$ as a common factor]

$= \frac{n(n+1)(2n+2)}{6}$

$= \frac{2n(n+1)(n+1)}{6}$ [writing $2n+2 = 2(n+1)$]

$= \frac{n(n+1)^2}{3}$

(b) Note that $\sum_{r=10}^{2p}\left(r^2 + \frac{1}{3}r\right)$ is a **partial sum**.

$\sum_{r=10}^{2p}\left(r^2 + \frac{1}{3}r\right) = \sum_{r=1}^{2p}\left(r^2 + \frac{1}{3}r\right) - \sum_{r=1}^{9}\left(r^2 + \frac{1}{3}r\right)$

$= S_{2p} - S_9$

$= \frac{2p(2p+1)^2}{3} - \frac{9(9+1)^2}{3}$ [using the result in (a)]

$= \frac{2p(2p+1)^2}{3} - \frac{9(10)^2}{3}$

$= \frac{2p(2p+1)^2}{3} - 300$