ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Proof by Induction

1. Prove by induction that, for all integers $n \geq 1$,

$$\sum_{r=1}^{n}(3r-1) = \frac{1}{2}n(3n+1).$$

First prove true for $n=1$, ie prove that $\sum_{r=1}^{1}(3r-1) = \frac{1}{2}(1)(3(1)+1)$.

LHS $= \sum_{r=1}^{1}(3r-1) = 3(1)-1 = 2$ and RHS $= \frac{1}{2}(1)(3(1)+1) = \frac{1}{2}(1)(4) = 2$

LHS = RHS, hence true for $n=1$.

Assume true for $n=k$, ie assume that $\sum_{r=1}^{k}(3r-1) = \frac{1}{2}k(3k+1)$.

Now prove also true for $n=k+1$, ie prove that

$$\sum_{r=1}^{k+1}(3r-1) = \frac{1}{2}(k+1)\big(3(k+1)+1\big) = \frac{1}{2}(k+1)(3k+4)$$

$$\begin{aligned}
\sum_{r=1}^{k+1}(3r-1) &= \sum_{r=1}^{k}(3r-1) + \big(3(k+1)-1\big) \\
&= \frac{1}{2}k(3k+1) + (3k+2) && \left[\text{since by assumption } \sum_{r=1}^{k}(3r-1) = \frac{1}{2}k(3k+1)\right] \\
&= \frac{1}{2}\big(k(3k+1) + 2(3k+2)\big) && \left[\text{removing } \frac{1}{2} \text{ as a factor}\right] \\
&= \frac{1}{2}(3k^2 + k + 6k + 4) \\
&= \frac{1}{2}(3k^2 + 7k + 4) \\
&= \frac{1}{2}(k+1)(3k+4)
\end{aligned}$$

Hence also true for $n=k+1$.

The statement is true for $n=1$ and if true for $n=k$, then it is also true for $n=k+1$.
Hence the statement is true for all integers $n \geq 1$ by induction.

2. Prove by induction that

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1), \qquad \forall n \in \mathbf{N}.$$

First prove true for $n=1$, ie prove that $\sum_{r=1}^{1} r(3r-1) = 1^2(1+1)$.

LHS $= \sum_{r=1}^{1} r(3r-1) = 1(3(1)-1) = 1(2) = 2$ and RHS $= 1^2(1+1) = 1(2) = 2$

LHS = RHS, hence true for $n=1$.

Assume true for $n=k$, ie assume that $\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$.

Now prove also true for $n=k+1$, ie prove that $\sum_{r=1}^{k+1} r(3r-1) = (k+1)^2((k+1)+1)$

$$= (k+1)^2(k+2)$$

$$\sum_{r=1}^{k+1} r(3r-1) = \sum_{r=1}^{k} r(3r-1) + (k+1)\big(3(k+1)-1\big)$$

$= k^2(k+1) + (k+1)(3k+2)$ [since by assumption $\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$]

$= (k+1)(k^2 + (3k+2))$ [removing $(k+1)$ as a common factor]

$= (k+1)(k^2 + 3k + 2)$

$= (k+1)(k+1)(k+2)$

$= (k+1)^2(k+2)$

Hence also true for $n=k+1$.

The statement is true for $n=1$ and if true for $n=k$, then it is also true for $n=k+1$.
Hence the statement is true $\forall n \in \mathbf{N}$ by induction.

3. Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n} 3^{r-1} = \frac{1}{2}(3^n - 1).$$

First prove true for $n = 1$, ie prove that $\sum_{r=1}^{1} 3^{r-1} = \frac{1}{2}(3^1 - 1)$.

LHS $= \sum_{r=1}^{1} 3^{r-1} = 3^{1-1} = 3^0 = 1$ and RHS $= \frac{1}{2}(3^1 - 1) = \frac{1}{2}(2) = 1$

LHS = RHS, hence true for $n = 1$.

Assume true for $n = k$, ie assume that $\sum_{r=1}^{k} 3^{r-1} = \frac{1}{2}(3^k - 1)$.

Now prove also true for $n = k + 1$, ie prove that $\sum_{r=1}^{k+1} 3^{r-1} = \frac{1}{2}(3^{k+1} - 1)$

$$\sum_{r=1}^{k+1} 3^{r-1} = \sum_{r=1}^{k} 3^{r-1} + 3^{(k+1)-1}$$

$= \frac{1}{2}(3^k - 1) + 3^k$ [since by assumption $\sum_{r=1}^{k} 3^{r-1} = \frac{1}{2}(3^k - 1)$]

$= \frac{1}{2}\left((3^k - 1) + 2(3^k)\right)$ [removing $\frac{1}{2}$ as a factor]

$= \frac{1}{2}(3(3^k) - 1)$

$= \frac{1}{2}(3^1 3^k - 1)$

$= \frac{1}{2}(3^{k+1} - 1)$

Hence also true for $n = k + 1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k + 1$.
Hence the statement is true for all positive integers *n* by induction.

4.(a) Prove by induction that, for all natural numbers $n \geq 1$,

$$\sum_{r=1}^{n} 3(r^2 - r) = (n-1)n(n+1).$$

First prove true for $n = 1$, ie prove that $\sum_{r=1}^{1} 3(r^2 - r) = (1-1)1(1+1)$.

LHS $= \sum_{r=1}^{1} 3(r^2 - r) = 3(1^2 - 1) = 3(0) = 0$ and RHS $= (1-1)1(1+1) = (0)(1)(2) = 0$

LHS = RHS, hence true for $n = 1$.

Assume true for $n = k$, ie assume that $\sum_{r=1}^{k} 3(r^2 - r) = (k-1)k(k+1)$.

Now prove also true for $n = k+1$, ie prove that $\sum_{r=1}^{k+1} 3(r^2 - r) = \big((k+1)-1\big)(k+1)\big((k+1)+1\big)$

$$= k(k+1)(k+2)$$

$$\begin{aligned}
\sum_{r=1}^{k+1} 3(r^2 - r) &= \sum_{r=1}^{k} 3(r^2 - r) + 3\big((k+1)^2 - (k+1)\big) \\
&= (k-1)k(k+1) + 3(k+1)^2 - 3(k+1) \\
&= (k+1)\big(k(k-1) + 3(k+1) - 3\big) \qquad \text{[removing } (k+1) \text{ as a common factor]} \\
&= (k+1)(k^2 - k + 3k + 3 - 3) \\
&= (k+1)(k^2 + 2k) \\
&= (k+1)k(k+2) \\
&= k(k+1)(k+2)
\end{aligned}$$

Hence also true for $n = k+1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k+1$.
Hence the statement is true for all natural numbers $n \geq 1$ by induction.

(b) Note that $\sum_{r=11}^{40} 3(r^2 - r)$ is a **partial sum**.

$$\sum_{r=11}^{40} 3(r^2 - r) = \sum_{r=1}^{40} 3(r^2 - r) - \sum_{r=1}^{10} 3(r^2 - r)$$

$$= (40-1)40(40+1) - (10-1)10(10+1) \qquad \text{[using the result in (a)]}$$

$$= (39)(40)(41) - (9)(10)(11)$$

$$= 63\ 960 - 990$$

$$= 62\ 970$$

5. Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n}(4r^3+3r^2+r)=n(n+1)^3\,.$$

First prove true for $n=1$, ie prove that $\sum_{r=1}^{1}(4r^3+3r^2+r)=1(1+1)^3$.

LHS $=\sum_{r=1}^{1}(4r^3+3r^2+r)=4(1)^3+3(1)^2+1=8$ and RHS $=1(1+1)^3=1(2)^3=8$

LHS = RHS, hence true for $n=1$.

Assume true for $n=k$, ie assume that $\sum_{r=1}^{k}(4r^3+3r^2+r)=k(k+1)^3$.

Now prove also true for $n=k+1$, ie prove that

$$\begin{aligned}\sum_{r=1}^{k+1}(4r^3+3r^2+r)&=(k+1)\big((k+1)+1\big)^3\\&=(k+1)(k+2)^3\end{aligned}$$

$$\begin{aligned}\sum_{r=1}^{k+1}(4r^3+3r^2+r)&=\sum_{r=1}^{k}(4r^3+3r^2+r)+\big(4(k+1)^3+3(k+1)^2+(k+1)\big)\\&=k(k+1)^3+4(k+1)^3+3(k+1)^2+(k+1)\\&=(k+1)\big(k(k+1)^2+4(k+1)^2+3(k+1)+1\big)\\&=(k+1)\big(k(k^2+2k+1)+4(k^2+2k+1)+3k+3+1\big)\\&=(k+1)(k^3+2k^2+k+4k^2+8k+4+3k+3+1)\\&=(k+1)(k^3+6k^2+12k+8)\qquad ...(*)\end{aligned}$$

To factorise the cubic expression $k^3+6k^2+12k+8$, find a factor using synthetic division.

	1	6	12	8
−2		−2	−8	−8
	1	4	4	0

remainder = 0, so $(k+2)$ is a factor $\Rightarrow$

$$\begin{aligned}k^3+6k^2+12k+8&=(k+2)(k^2+4k+4)\\&=(k+2)(k+2)(k+2)\\&=(k+2)^3\end{aligned}$$

Hence from (*), $\sum_{r=1}^{k+1}(4r^3+3r^2+r)=(k+1)(k+2)^3$ and so also true for $n=k+1$.

The statement is true for $n=1$ and if true for $n=k$, then it is also true for $n=k+1$. Hence the statement is true for all positive integers n by induction.

Note

When attempting to factorise $k^3+6k^2+12k+8$, you can see that the answer should be $(k+2)^3$ by comparing with the expression that you want to prove, ie $(k+1)(k+2)^3$.

6. Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}.$$

First prove true for $n=1$, ie prove that $\sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1+1}$.

$$\text{LHS} = \sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1(1+1)} = \frac{1}{1(2)} = \frac{1}{2} \quad \text{and} \quad \text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

LHS = RHS, hence true for $n=1$.

Assume true for $n=k$, ie assume that $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$.

Now prove also true for $n=k+1$, ie prove that $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)\big((k+1)+1\big)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{[since by assumption } \sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}\text{]}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \qquad \text{[making a common denominator]}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} \qquad \text{[cancelling the factor of } (k+1)\text{]}$$

Hence also true for $n=k+1$.

The statement is true for $n=1$ and if true for $n=k$, then it is also true for $n=k+1$.
Hence the statement is true for all positive integers *n* by induction.

7. Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.$$

First prove true for $n = 1$, ie prove that $\sum_{r=1}^{1} \frac{1}{r(r+1)} = 1 - \frac{1}{1+1}$.

LHS $= \sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1(1+1)} = \frac{1}{1(2)} = \frac{1}{2}$ and RHS $= 1 - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$

LHS = RHS, hence true for $n = 1$.

Assume true for $n = k$, ie assume that $\sum_{r=1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}$.

Now prove also true for $n = k+1$, ie prove that $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = 1 - \frac{1}{(k+1)+1}$

$$= 1 - \frac{1}{k+2}$$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)((k+1)+1)}$$

$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$ [since by assumption $\sum_{r=1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}$]

$$= 1 - \left(\frac{1}{k+1} - \frac{1}{(k+1)(k+2)} \right)$$

$= 1 - \left(\frac{k+2}{(k+1)(k+2)} - \frac{1}{(k+1)(k+2)} \right)$ [making a common denominator]

$$= 1 - \left(\frac{k+2-1}{(k+1)(k+2)} \right)$$

$$= 1 - \left(\frac{k+1}{(k+1)(k+2)} \right)$$

$= 1 - \frac{1}{k+2}$ [cancelling the factor of $(k+1)$]

Hence also true for $n = k+1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k+1$. Hence the statement is true for all positive integers *n* by induction.

8. Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

First prove true for $n=1$, ie prove that $\sum_{r=1}^{1} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(1+1)(1+2)}$.

$$\text{LHS} = \sum_{r=1}^{1} \frac{1}{r(r+1)(r+2)} = \frac{1}{1(1+1)(1+2)} = \frac{1}{1(2)(3)} = \frac{1}{6}$$

$$\text{RHS} = \frac{1}{4} - \frac{1}{2(1+1)(1+2)} = \frac{1}{4} - \frac{1}{2(2)(3)} = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

LHS = RHS, hence true for $n=1$.

Assume true for $n=k$, ie assume that $\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$.

Now prove also true for $n=k+1$,

ie prove that $\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2\big((k+1)+1\big)\big((k+1)+2\big)}$

$$= \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$

$$\begin{aligned}
\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} &= \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} + \frac{1}{(k+1)\big((k+1)+1\big)\big((k+1)+2\big)} \\
&= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\
&= \frac{1}{4} - \left(\frac{1}{2(k+1)(k+2)} - \frac{1}{(k+1)(k+2)(k+3)} \right) \\
&= \frac{1}{4} - \left(\frac{k+3}{2(k+1)(k+2)(k+3)} - \frac{2}{(k+1)(k+2)(k+3)} \right) \\
&= \frac{1}{4} - \left(\frac{k+3-2}{2(k+1)(k+2)(k+3)} \right) \\
&= \frac{1}{4} - \left(\frac{k+1}{2(k+1)(k+2)(k+3)} \right) \\
&= \frac{1}{4} - \frac{1}{2(k+2)(k+3)}
\end{aligned}$$

[cancelling the factor of $(k+1)$]

Hence also true for $n=k+1$.

The statement is true for $n=1$ and if true for $n=k$, then it is also true for $n=k+1$.
Hence the statement is true for all positive integers *n* by induction.