ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Number Theory and Methods of Proof

1. First divide 14654 by 1326: $14654 = 11 \times 1326 + 68$

Then divide 1326 by 68: $1326 = 19 \times 68 + 34$ ...(*)

Then divide 68 by 34: $68 = 2 \times 34 + 0$

The greatest common divisor (gcd) is the last non-zero remainder.
Hence from (*), gcd = 34.

To express the gcd in the form $1326a + 14654b$ for integers *p* and *q*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*): $34 = 1326 - 19(68)$ [now substitute for the remainder 68]
$= 1326 - 19\big(14654 - 11(1326)\big)$ [now multiply out the brackets]
$= 1326 - 19(14654) + 209(1326)$ [now collect like terms to simplify]
$= 210(1326) - 19(14654)$

Hence $34 = 210(1326) - 19(14654)$ and so $\text{gcd} = 1326a + 14654b$ where $a = 210$ and $b = -19$.

2. Use the Euclidean algorithm to find the greatest common divisor (gcd) of 306 and 119.

First divide 306 by 119: $306 = 2 \times 119 + 68$

Then divide 119 by 68: $119 = 1 \times 68 + 51$

Then divide 68 by 51: $68 = 1 \times 51 + 17$ ...(*)

Then divide 51 by 17: $51 = 3 \times 17 + 0$

The greatest common divisor (gcd) is the last non-zero remainder.
Hence from (*), gcd = 17.

To express the gcd in the form $306a + 119b$ for integers *a* and *b*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*): $17 = 68 - 1(51)$ [now substitute for the remainder 51]

$= 68 - 1(119 - 1(68))$ [now multiply out the brackets]

$= 68 - 1(119) + 1(68)$ [now collect like terms to simplify]

$= 2(68) - 1(119)$ [now substitute for the remainder 68]

$= 2(306 - 2(119)) - 1(119)$ [now multiply out the brackets]

$= 2(306) - 4(119) - 1(119)$ [now collect like terms to simplify]

$= 2(306) - 5(119)$

Hence $17 = 2(306) - 5(119)$ and so $306a + 119b = 17$ where $a = 2$ and $b = -5$.

3. Use the Euclidean algorithm to find the greatest common divisor (gcd) of 1595 and 1218.

First divide 1595 by 1218: $1595 = 1 \times 1218 + 377$

Then divide 1218 by 377: $1218 = 3 \times 377 + 87$

Then divide 377 by 87: $377 = 4 \times 87 + 29$ ...(*)

Then divide 87 by 29: $87 = 3 \times 29 + 0$

The greatest common divisor (gcd) is the last non-zero remainder.
Hence from (*), gcd = 29.

To express the gcd in the form $1595a + 1218b$ for integers *a* and *b*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*): $29 = 377 - 4(87)$ [now substitute for the remainder 87]

$= 377 - 4\left(1218 - 3(377)\right)$ [now multiply out the brackets]

$= 377 - 4(1218) + 12(377)$ [now collect like terms to simplify]

$= 13(377) - 4(1218)$ [now substitute for the remainder 377]

$= 13\left(1595 - 1(1218)\right) - 4(1218)$ [now multiply out the brackets]

$= 13(1595) - 13(1218) - 4(1218)$ [now collect like terms to simplify]

$= 13(1595) - 17(1218)$

Hence $29 = 13(1595) - 17(1218)$ and so $1595a + 1218b = 29$ where $a = 13$ and $b = -17$.

4. First divide 1204 by 833: $1204 = 1 \times 833 + 371$

Then divide 833 by 371: $833 = 2 \times 371 + 91$

Then divide 371 by 91: $371 = 4 \times 91 + 7$ ...(*)

Then divide 91 by 7: $91 = 13 \times 7 + 0$

The greatest common divisor (gcd) is the last non-zero remainder.
Hence from (*), gcd = 7.

To express the gcd in the form $1204a + 833b$ for integers *a* and *b*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*): $7 = 371 - 4(91)$ [now substitute for the remainder 91]

$= 371 - 4\big(833 - 2(371)\big)$ [now multiply out the brackets]

$= 371 - 4(833) + 8(371)$ [now collect like terms to simplify]

$= 9(371) - 4(833)$ [now substitute for the remainder 371]

$= 9\big(1204 - 1(833)\big) - 4(833)$ [now multiply out the brackets]

$= 9(1204) - 9(833) - 4(833)$ [now collect like terms to simplify]

$= 9(1204) - 13(833)$

Hence $7 = 9(1204) - 13(833)$ and so $\text{gcd} = 1204a + 833b$ where $a = 9$ and $b = -13$.

5. Use the Euclidean algorithm to find the greatest common divisor (gcd) of 599 and 53.

First divide 599 by 53: $599 = 11 \times 53 + 16$

Then divide 53 by 16: $53 = 3 \times 16 + 5$

Then divide 16 by 5: $16 = 3 \times 5 + 1$ ...(*)

Then divide 5 by 1: $5 = 5 \times 1 + 0$

The greatest common divisor (gcd) is the last non-zero remainder.
Hence from (*), gcd = 1.

To express the gcd in the form $599p + 53q$ for integers *p* and *q*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*): $1 = 16 - 3(5)$ [now substitute for the remainder 5]

$= 16 - 3\left(53 - 3(16)\right)$ [now multiply out the brackets]

$= 16 - 3(53) + 9(16)$ [now collect like terms to simplify]

$= 10(16) - 3(53)$ [now substitute for the remainder 16]

$= 10\left(599 - 11(53)\right) - 3(53)$ [now multiply out the brackets]

$= 10(599) - 110(53) - 3(53)$ [now collect like terms to simplify]

$= 10(599) - 113(53)$

Hence $1 = 10(599) - 113(53)$ and so $599p + 53q = 1$ where $p = 10$ and $q = -113$.

6. Use the Euclidean algorithm to find the greatest common divisor (gcd) of 149 and 139.

First divide 149 by 139: $149 = 1 \times 139 + 10$

Then divide 139 by 10: $139 = 13 \times 10 + 9$

Then divide 10 by 9: $10 = 1 \times 9 + 1$ ...(*)

Then divide 9 by 1: $9 = 9 \times 1 + 0$

The greatest common divisor (gcd) is the last non-zero remainder.
Hence from (*), gcd = 1.

To express the gcd in the form $149x + 139y$ for integers *x* and *y*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*): $1 = 10 - 1(9)$ [now substitute for the remainder 9]

$= 10 - 1\big(139 - 13(10)\big)$ [now multiply out the brackets]

$= 10 - 1(139) + 13(10)$ [now collect like terms to simplify]

$= 14(10) - 1(139)$ [now substitute for the remainder 10]

$= 14\big(149 - 1(139)\big) - 1(139)$ [now multiply out the brackets]

$= 14(149) - 14(139) - 1(139)$ [now collect like terms to simplify]

$= 14(149) - 15(139)$

Hence $1 = 14(149) - 15(139)$ and so $149x + 139y = 1$ where $x = 14$ and $y = -15$.

7. Use the Euclidean algorithm to find the greatest common divisor (gcd) of 3066 and 713.

First divide 3066 by 713: $3066 = 4 \times 713 + 214$

Then divide 713 by 214: $713 = 3 \times 214 + 71$

Then divide 214 by 71: $214 = 3 \times 71 + 1$ ...(*)

Then divide 71 by 1: $71 = 71 \times 1 + 0$

The greatest common divisor (gcd) is the last non-zero remainder.
Hence from (*), gcd = 1.

To express the gcd in the form $3066p + 713q$ for integers *p* and *q*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*): $1 = 214 - 3(71)$ [now substitute for the remainder 71]
$= 214 - 3\big(713 - 3(214)\big)$ [now multiply out the brackets]
$= 214 - 3(713) + 9(214)$ [now collect like terms to simplify]
$= 10(214) - 3(713)$ [now substitute for the remainder 214]
$= 10\big(3066 - 4(713)\big) - 3(713)$ [now multiply out the brackets]
$= 10(3066) - 40(713) - 3(713)$ [now collect like terms to simplify]
$= 10(3066) - 43(713)$

Hence $1 = 10(3066) - 43(713)$ and so $3066p + 713q = 1$ where $p = 10$ and $q = -43$.

8. First divide 729 by 487: 729 = 1 × 487 + 242

Then divide 487 by 242: 487 = 2 × 242 + 3

Then divide 242 by 3: 242 = 80 × 3 + 2

Then divide 3 by 2: 3 = 1 × 2 + 1 ...(*)

Then divide 2 by 1: 2 = 2 × 1 + 0

The greatest common divisor (gcd) is the last non-zero remainder.
Hence from (*), gcd = 1.

To express the gcd in the form $487x + 729y$ for integers *x* and *y*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*):	$1 = 3 - 1(2)$	[now substitute for the remainder 2]
	$= 3 - 1\big(242 - 80(3)\big)$	[now multiply out the brackets]
	$= 3 - 1(242) + 80(3)$	[now collect like terms to simplify]
	$= 81(3) - 1(242)$	[now substitute for the remainder 3]
	$= 81\big(487 - 2(242)\big) - 1(242)$	[now multiply out the brackets]
	$= 81(487) - 162(242) - 1(242)$	[now collect like terms to simplify]
	$= 81(487) - 163(242)$	[now substitute for the remainder 242]
	$= 81(487) - 163\big(729 - 1(487)\big)$	[now multiply out the brackets]
	$= 81(487) - 163(729) + 163(487)$	[now collect like terms]
	$= 244(487) - 163(729)$	

Hence $1 = 244(487) - 163(729)$ and so $487x + 729y = 1$ where $x = 244$ and $y = -163$.

9. First divide 231 by 17: $231 = 13 \times 17 + 10$

Then divide 17 by 10: $17 = 1 \times 10 + 7$

Then divide 10 by 7: $10 = 1 \times 7 + 3$

Then divide 7 by 3: $7 = 2 \times 3 + 1$ ...(*)

Then divide 3 by 1: $3 = 3 \times 1 + 0$

The highest common factor (hcf) is the last non-zero remainder.
Hence from (*), hcf = 1 and (231, 17) = 1.

To express the hcf in the form $231x + 17y$ for integers *x* and *y*, start at line (*) and work backwards up the lines replacing each remainder and then simplifying.

From (*):	$1 = 7 - 2(3)$	[now substitute for the remainder 3]
	$= 7 - 2\big(10 - 1(7)\big)$	[now multiply out the brackets]
	$= 7 - 2(10) + 2(7)$	[now collect like terms to simplify]
	$= 3(7) - 2(10)$	[now substitute for the remainder 7]
	$= 3\big(17 - 1(10)\big) - 2(10)$	[now multiply out the brackets]
	$= 3(17) - 3(10) - 2(10)$	[now collect like terms to simplify]
	$= 3(17) - 5(10)$	[now substitute for the remainder 10]
	$= 3(17) - 5\big(231 - 13(17)\big)$	[now multiply out the brackets]
	$= 3(17) - 5(231) + 65(17)$	[now collect like terms to simplify]
	$= 68(17) - 5(231)$	

Hence $1 = 68(17) - 5(231)$ and so $231x + 17y = 1$ where $x = -5$ and $y = 68$.

<u>Note</u>

The **highest common factor** (hcf) of 231 and 17 is the same as the **greatest common divisor** (gcd) of 231 and 17.

10. To express 1234_{10} in base 7, keep dividing by 7 using the division algorithm.

First divide 1234 by 7: 1234 = 7 × 176 + **2**

Then divide 176 by 7: 176 = 7 × 25 + **1**

Then divide 25 by 7: 25 = 7 × 3 + **4**

Then divide 3 by 7: 3 = 7 × 0 + **3**

The answer can now be read from the remainders in reverse order: $1234_{10} = 3412_7$

Check: $3412_7 = (3\times 7^3) + (4\times 7^2) + (1\times 7^1) + (2\times 7^0) = 1234$ √

11.(a) Three consecutive integers are *n*, $n+1$ and $n+2$ for some integer *n*.
Let *S* denote the sum of the three consecutive integers *n*, $n+1$ and $n+2$.

$S = n + (n+1) + (n+2) = n + n + 1 + n + 2 = 3n + 3 = 3(n+1) = 3p$ where $p = n+1$

$S = 3p$ for some integer *p*, hence the sum of any three consecutive integers is divisible by 3.

(b) Let *a* be an odd integer. Then $a = 2k+1$ for some integer *k*.

$a = 2k + 1 = k + (k+1)$

Since *k* and $k+1$ are consecutive integers, this means that any odd integer can be expressed as the sum of two consecutive integers.

12. We will use a **direct proof**.

Two consecutive odd numbers are $a = 2k+1$ and $b = (2k+1) + 2 = 2k + 3$ for some integer *k*.

The difference between the squares of the two consecutive odd numbers is $b^2 - a^2$.

$$
\begin{aligned}
b^2 - a^2 = (2k+3)^2 - (2k+1)^2 &= (4k^2 + 12k + 9) - (4k^2 + 4k + 1) \\
&= 4k^2 + 12k + 9 - 4k^2 - 4k - 1 \\
&= 8k + 8 \\
&= 8(k+1) \\
&= 8p \qquad \text{where } p = k+1
\end{aligned}
$$

$b^2 - a^2 = 8p$ for some integer *p*, hence the difference between the squares of two consecutive odd numbers is divisible by 8.

13. Consider statement A: If n is a multiple of 9 then so is n^2.

Statement A is **true** and we can use a **direct proof** to prove this.

Let n be a multiple of 9. Then $n = 9k$ for some integer k.

$n^2 = (9k)^2 = 81k^2 = 9(9k^2) = 9p$ where $p = 9k^2$

$n^2 = 9p$ for some integer p, hence n^2 is a multiple of 9 when n is a multiple of 9.

Now consider statement B: If n^2 is a multiple of 9 then so is n.

Statement B is **false** and we can prove this by finding a **counterexample**.

When $n = 3$: $n^2 = 3^2 = 9 = 9(1)$, hence n^2 is a multiple of 9, however n itself is not a multiple of 9.

Hence $n = 3$ is a counterexample which proves that statement B is false.

Note

There are other possible counterexamples, eg $n = 6$, $n = 12$, $n = 15$, ...

14. Consider statement A: If a positive integer *p* is prime, then so is $2p+1$.

Statement A is **false** and we can prove this by finding a **counterexample**.

Consider the prime number $p=7$.
Then $2p+1=2(7)+1=15$ and $15=3\times 5$, so 15 is not prime.

Hence $p=7$ is a counterexample which proves that statement A is false.

Now consider statement B: If a positive integer *n* has remainder 1 when divided by 3, then n^3 also has remainder 1 when divided by 3.

Statement B is **true** and we can use a **direct proof** to prove this.

Let the positive integer *n* have remainder 1 when divided by 3.
Then $n=3k+1$ for some integer *k*.

$$
\begin{aligned}
n^3=(3k+1)^3 &= \binom{3}{0}(3k)^3 1^0+\binom{3}{1}(3k)^2 1^1+\binom{3}{2}(3k)^1 1^2+\binom{3}{3}(3k)^0 1^3 \\
&= 27k^3+3(9k^2)+3(3k)+1 \\
&= 27k^3+27k^2+9k+1 \\
&= 3(9k^3+9k^2+3k)+1 \\
&= 3p+1 \qquad \text{where } p=9k^3+9k^2+3k
\end{aligned}
$$

$n^3=3p+1$ for some integer *p*, hence n^3 also has remainder 1 when divided by 3.

<u>Notes</u>

(1) There are other possible counterexamples which can be used to prove that statement A is false,, eg $p=13$, $n=17$, $n=19$, ...

(2) $(3k+1)^3$ can be expanded without using the binomial theorem as follows.

$$
\begin{aligned}
(3k+1)^3=(3k+1)(3k+1)^2 &= (3k+1)(9k^2+6k+1) \\
&= 3k(9k^2+6k+1)+1(9k^2+6k+1) \\
&= 27k^3+18k^2+3k+9k^2+6k+1 \\
&= 27k^3+27k^2+9k+1
\end{aligned}
$$

15. Consider statement A: For all natural numbers *m*, if m^2 is divisible by 4, then *m* is divisible by 4.

Statement A is **false** and we can prove this by finding a **counterexample**.

When $m = 2$: $m^2 = 2^2 = 4 = 4(1)$, hence m^2 is divisible by 4, however *m* itself is not divisible by 4.

Hence $m = 2$ is a counterexample which proves that statement B is false.

Now consider statement B: The cube of any odd integer *p* plus the square of any even integer *q* is always odd.

Statement B is **true** and we can use a **direct proof** to prove this.

Let *p* be an odd integer. Then $p = 2k + 1$ for some integer *k*.

$$p^3 = (2k+1)^3 = \binom{3}{0}(2k)^3 1^0 + \binom{3}{1}(2k)^2 1^1 + \binom{3}{2}(2k)^1 1^2 + \binom{3}{3}(2k)^0 1^3$$
$$= 8k^3 + 3(4k^2) + 3(2k) + 1$$
$$= 8k^3 + 12k^2 + 6k + 1$$

Let *q* be an even integer. Then $q = 2m$ for some integer *m*.

$$q^2 = (2m)^2 = 4m^2$$

Then $p^3 + q^2 = (8k^3 + 12k^2 + 6k + 1) + 4m^2$
$$= 8k^3 + 12k^2 + 6k + 4m^2 + 1$$
$$= 2(4k^3 + 6k^2 + 3k + 2m^2) + 1$$
$$= 2a + 1 \qquad \text{where } a = 4k^3 + 6k^2 + 3k + 2m^2$$

$p^3 + q^2 = 2a + 1$ for some integer *a*, hence $p^3 + q^2$ is odd.

Notes

(1) There are other possible counterexamples which can be used to prove that statement A is false,, eg $m=6$, $n=10$, $n=14$, ...

(2) $(2k+1)^3$ can be expanded without using the binomial theorem as follows.

$$\begin{aligned}(2k+1)^3 = (2k+1)(2k+1)^2 &= (2k+1)(4k^2+4k+1)\\ &= 2k(4k^2+4k+1)+1(4k^2+4k+1)\\ &= 8k^3+8k^2+2k+4k^2+4k+1\\ &= 8k^3+12k^2+6k+1\end{aligned}$$

16. Consider statement A: $p(n)$ is always even

Statement A is **true** and there are two ways of proving this.

Method 1

We can prove that statement A is true by factorising $p(n) = n^2 + n$.

$p(n) = n^2 + n = n(n+1)$

This expression shows that $p(n)$ is the product of two consecutive integers, one of which will be even and one of which will be odd. Hence $p(n)$ is always even since the product of an even integer and an odd integer is even.

Method 2

We can use a **direct proof** to prove that $p(n)$ is even for all positive integers *n*.
We will consider the cases when *n* is even and *n* is odd separately.

Let *n* be even. Then $n = 2k$ for some integer *k*.

$p(n) = n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k) = 2p$ where $p = 2k^2 + k$

$p(n) = 2p$ for some integer *p* and hence $p(n)$ is even when *n* is even.

Now let *n* be odd. Then $n = 2k + 1$ for some integer *k*.

$$\begin{aligned} p(n) = n^2 + n = (2k+1)^2 + (2k+1) &= 4k^2 + 4k + 1 + 2k + 1 \\ &= 4k^2 + 6k + 2 \\ &= 2(2k^2 + 3k + 1) \\ &= 2p \qquad \text{where } p = 2k^2 + 3k + 1 \end{aligned}$$

$p(n) = 2p$ for some integer *p* and hence $p(n)$ is even when *n* is odd.

We have proved that $p(n)$ is even when *n* is even and $p(n)$ is also even when *n* is odd.
This means that $p(n)$ is even for all positive integers *n*.

Now consider statement B: $p(n)$ is always a multiple of 3

Statement B is **false** and we can prove this by finding a **counterexample**.

$p(1) = 1^2 + 1 = 2$, hence $p(1)$ is not a multiple of 3.

Hence $n = 1$ is a counterexample which proves that statement B is false.

Note

There are other possible counterexamples, eg $n = 4$, $n = 7$, $n = 10$, ...

17.(a) Consider the statement that $n^3 - n$ is always divisible by 6.

This statement is **true** and we can prove this by fully factorising $n^3 - n$.

$n^3 - n = n(n^2 - 1) = n(n-1)(n+1) = (n-1)n(n+1)$

This expression shows that $n^3 - n$ is the product of the three consecutive integers $n-1$, n and $n+1$, at least one of which will be even and one of which will be divisible by 3. Hence $n^3 - n$ is always divisible by 6 since it is known that the product of three consecutive integers is divisible by 6

(b) Consider the statement that $n^3 + n + 5$ is always prime.

This statement is **false** and we can prove this by finding a **counterexample**.

When $n = 2$: $n^3 + n + 5 = 2^3 + 2 + 5 = 15$ and $15 = 3 \times 5$, so 15 is not prime

Hence $n^3 + n + 5$ is not prime when $n = 2$, thus $n = 2$ is a counterexample which proves that the statement is false.

Note

There are other possible counterexamples, eg $n = 3$, $n = 5$, ...

18. Let x be an irrational number and assume that $2+x$ is rational.

$2+x$ is rational, so $2+x=\frac{m}{n}$ for some integers m and n where $n \neq 0$.

$$2+x=\frac{m}{n} \quad \Rightarrow \quad x=\frac{m}{n}-2=\frac{m}{n}-\frac{2}{1}=\frac{m}{n}-\frac{2n}{n}=\frac{m-2n}{n}=\frac{p}{q}$$ where $p=m-2n$ and $q=n$

Hence $x=\frac{p}{q}$ for some integers p and q where $q \neq 0$.

This means that x is a rational number which is a contradiction.
Hence if x is an irrational number, then $2+x$ is irrational.

19.(a) The negation of the statement 'm is even or n is even' is 'both m and n are odd'.

(b) The contrapositive of the statement 'if mn is even then m is even or n is even' is 'if both m and n are odd then mn is odd'.

Let both m and n be odd.
Then $m=2k+1$ and $n=2p+1$ for some integers k and p.

$$\begin{aligned} mn=(2k+1)(2p+1) &= 4kp+2k+2p+1 \\ &= 2(2kp+k+p)+1 \\ &= 2a+1 \end{aligned}$$ where $a=2kp+k+p$

$mn=2a+1$ for some integer a, hence mn is odd.

We have proved that the contrapositive statement is true, thus we have proved that the original statement is also true, ie if mn is even then m is even or n is even.

<u>Note</u>

The contrapositive of the statement 'if A, then B' is 'if not B, then not A'. The statement and its contrapositive are logically equivalent, so the statement is true if its contrapositive is true (and vice versa).

20. The contrapositive of the statement ‘if n^2 is even, then *n* is even’ is ‘if *n* is odd, then n^2 is odd’.

Let *n* be an odd integer. Then $n = 2k + 1$ for some integer k.

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2p + 1$ where $p = 2k^2 + 2k$

$n^2 = 2p$ for some integer *p*, hence n^2 is odd.

We have proved that the contrapositive statement is true, thus we have proved that the original statement is also true, ie if n^2 is even, then *n* is even.

<u>Note</u>

The contrapositive of the statement ‘if A, then B’ is ‘if not B, then not A’. The statement and its contrapositive are logically equivalent, so the statement is true if its contrapositive is true (and vice versa).

21. Prove by induction that, for all positive integers *n*,

$$A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

First prove true for $n = 1$, ie prove that $A^1 = \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix}$.

$$\text{LHS} = A^1 = A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \text{RHS} = \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

LHS = RHS, hence true for $n = 1$.

Assume true for $n = k$, ie assume that $A^k = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}$.

Now prove also true for $n = k+1$, ie prove that

$$A^{k+1} = \begin{pmatrix} (k+1)+1 & k+1 \\ -(k+1) & 1-(k+1) \end{pmatrix} = \begin{pmatrix} k+2 & k+1 \\ -k-1 & -k \end{pmatrix}$$

$$A^{k+1} = A^k A^1 = A^k A = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \qquad \left[\text{since by assumption } A^k = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}\right]$$

$$= \begin{pmatrix} 2(k+1)+(-1)k & 1(k+1)+0k \\ 2(-k)+(-1)(1-k) & 1(-k)+0(1-k) \end{pmatrix}$$

$$= \begin{pmatrix} 2k+2-k & k+1 \\ -2k-1+k & -k \end{pmatrix}$$

$$= \begin{pmatrix} k+2 & k+1 \\ -k-1 & -k \end{pmatrix}$$

Hence also true for $n = k+1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k+1$.
Hence the statement is true for all positive integers *n* by induction.

22. Prove by induction that, for all positive integers $n \geq 1$,

$$A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, \quad \text{where } \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}.$$

First prove true for $n = 1$, ie prove that $A^1 = \begin{pmatrix} 2^1 & a(2^1 - 1) \\ 0 & 1 \end{pmatrix}$.

$$\text{LHS} = A^1 = A = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \text{RHS} = \begin{pmatrix} 2^1 & a(2^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & a(1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$$

LHS = RHS, hence true for $n = 1$.

Assume true for $n = k$, ie assume that $A^k = \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix}$.

Now prove also true for $n = k + 1$, ie prove that $A^{k+1} = \begin{pmatrix} 2^{k+1} & a(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$

$$A^{k+1} = A^k A^1 = A^k A = \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} \quad \left[\text{since by assumption } A^k = \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 2^k 2 + a(2^k - 1)(0) & 2^k(a) + a(2^k - 1)(1) \\ 0(2) + 1(0) & 0(a) + 1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 2^k 2 & a2^k + a(2^k - 1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^k 2^1 & a2^k + a2^k - a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 2a2^k - a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & a2^k 2^1 - a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & a2^{k+1} - a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & a(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$$

Hence also true for $n = k + 1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k + 1$.
Hence the statement is true for all positive integers $n \geq 1$ by induction.

Note

The working to prove true for $n = k+1$ could also be organised as follows:

$$A^{k+1} = A^k A^1 = A^k A = \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} \text{ [since by assumption } A^k = \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix}\text{]}$$

$$= \begin{pmatrix} 2^k 2 + a(2^k - 1)(0) & 2^k(a) + a(2^k - 1)(1) \\ 0(2) + 1(0) & 0(a) + 1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 2^k 2 & a2^k + a(2^k - 1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^k 2^1 & a\left(2^k + (2^k - 1)\right) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & a\left(2(2^k) - 1\right) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & a(2^1 2^k - 1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & a(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$$

23. Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers *n*.

First prove true for $n = 1$, ie prove that $4^1 - 1$ is divisible by 3.

$4^1 - 1 = 3 = 3(1)$, so $4^1 - 1$ is divisible by 3 and hence true for $n = 1$.

Assume true for $n = k$, ie assume that $4^k - 1$ is divisible by 3.
Then $4^k - 1 = 3p$ for some integer *p* $\Rightarrow$ $4^k = 3p + 1$ for some integer *p*

Now prove also true for $n = k + 1$, ie prove that $4^{k+1} - 1$ is divisible by 3.

$$
\begin{aligned}
4^{k+1} - 1 = 4^k 4^1 - 1 &= 4(4^k) - 1 \\
&= 4(3p + 1) - 1 \qquad \text{[since by assumption } 4^k = 3p + 1 \text{ for some } p \in \mathbf{Z}] \\
&= 12p + 4 - 1 \\
&= 12p + 3 \\
&= 3(4p + 1) \\
&= 3m \qquad \text{where } m = 4p + 1
\end{aligned}
$$

$4^{k+1} - 1 = 3m$ for some integer *m*, so $4^{k+1} - 1$ is divisible by 3 and hence also true for $n = k + 1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k + 1$.
Hence the statement is true for all positive integers *n* by induction.

24. Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 for all integers $n \geq 2$.

First prove true for $n = 2$, ie prove that $8^2 + 3^{2-2}$ is divisible by 5.

$8^2 + 3^{2-2} = 8^2 + 3^0 = 64 + 1 = 65 = 5(13)$, so $8^2 + 3^{2-2}$ is divisible by 5 and hence true for $n = 1$.

Assume true for $n = k$, ie assume that $8^k + 3^{k-2}$ is divisible by 5.
Then $8^k + 3^{k-2} = 5p$ for some integer p $\Rightarrow$ $8^k = 5p - 3^{k-2}$ for some integer p

Now prove also true for $n = k + 1$, ie prove that $8^{k+1} + 3^{(k+1)-2}$ is divisible by 5.

$$\begin{aligned}
8^{k+1} + 3^{(k+1)-2} &= 8^{k+1} + 3^{k-1} \\
&= 8^k 8^1 + 3^{k-2} 3^1 \\
&= 8(8^k) + 3(3^{k-2}) \\
&= 8(5p - 3^{k-2}) + 3(3^{k-2}) \quad \text{[since by assumption } 8^k = 5p - 3^{k-2} \text{ for some } p \in \mathbf{Z}] \\
&= 40p - 8(3^{k-2}) + 3(3^{k-2}) \\
&= 40p - 5(3^{k-2}) \\
&= 5(8p - 3^{k-2})
\end{aligned}$$

$8^{k+1} + 3^{(k+1)-2} = 5m$ for some integer m, so $8^{k+1} + 3^{(k+1)-2}$ is divisible by 5 and hence also true for $n = k + 1$.

The statement is true for $n = 2$ and if true for $n = k$, then it is also true for $n = k + 1$.
Hence the statement is true for all integers $n \geq 2$ by induction.

<u>Note</u>

The basis for the proof by induction in this question is $n = 2$.

25. Prove by induction that $\dfrac{d^n}{dx^n}(xe^x) = (x+n)e^x$ for all integers $n \geq 1$.

First prove true for $n = 1$, ie prove that $\dfrac{d^1}{dx^1}(xe^x) = (x+1)e^x$.

$$\text{LHS} = \frac{d^1}{dx^1}(xe^x) = \frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(1) \qquad \text{[using the \textbf{product rule}]}$$
$$= xe^x + e^x(1)$$
$$= xe^x + e^x$$
$$= e^x(x+1)$$
$$= (x+1)e^x$$
$$= \text{RHS}$$

LHS = RHS, hence true for $n = 1$.

Assume true for $n = k$, ie assume that $\dfrac{d^k}{dx^k}(xe^x) = (x+k)e^x$.

Now prove also true for $n = k+1$, ie prove that $\dfrac{d^{k+1}}{dx^{k+1}}(xe^x) = (x+k+1)e^x$.

$$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^x)\right)$$
$$= \frac{d}{dx}\big((x+k)e^x\big) \qquad \text{[since by assumption } \frac{d^k}{dx^k}(xe^x) = (x+k)e^x \text{]}$$
$$= (x+k)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+k) \qquad \text{[using the \textbf{product rule}]}$$
$$= (x+k)e^x + e^x(1)$$
$$= (x+k)e^x + e^x$$
$$= e^x\big((x+k)+1\big)$$
$$= (x+k+1)e^x$$

Hence also true for $n = k+1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k+1$.
Hence the statement is true for all integers $n \geq 1$ by induction.

Note

$\frac{d^n}{dx^n}(xe^x)$ is the n^{th} derivative of xe^x, ie the function obtained by differentiating xe^x n times.

$\frac{d^{k+1}}{dx^{k+1}}(xe^x)$ is the function obtained by differentiating xe^x $(k+1)$ times and $\frac{d^k}{dx^k}(xe^x)$ is the function obtained by differentiating xe^x k times

To find $\frac{d^{k+1}}{dx^{k+1}}(xe^x)$, you just need to differentiate $\frac{d^k}{dx^k}(xe^x)$.

Hence $\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^x)\right)$.

26. Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$.

First prove true for $n = 1$, ie prove that $(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$.

LHS $= (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$ and RHS $= \cos 1\theta + i\sin 1\theta = \cos\theta + i\sin\theta$

LHS = RHS, hence true for $n = 1$.

Assume true for $n = k$, ie assume that $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

Now prove also true for $n = k + 1$,
ie prove that $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$.

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1 \\
&= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \qquad \text{[by assumption]} \\
&= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta + i^2\sin k\theta\sin\theta \\
&= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta - \sin k\theta\sin\theta \\
&= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta) \\
&= \cos(k\theta + \theta) + i\sin(k\theta + \theta) \\
&= \cos(k+1)\theta + i\sin(k+1)\theta
\end{aligned}$$

Hence also true for $n = k + 1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k + 1$.
Hence the statement is true for all integers $n \geq 1$ by induction.

27. Prove by induction that for $a > 0$,

$$(1+a)^n \geq 1+na$$

for all positive integers *n*.

First prove true for $n=1$, ie prove that $(1+a)^1 \geq 1+1a$.

$(1+a)^1 = 1+a = 1+1a \quad \Rightarrow \quad (1+a)^1 \geq 1+1a$, hence true for $n=1$.

Assume true for $n=k$, ie assume that $(1+a)^k \geq 1+ka$.

Now prove also true for $n=k+1$, ie prove that $(1+a)^{k+1} \geq 1+(k+1)a$.

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k(1+a)^1 \\ &\geq (1+ka)(1+a) && \text{[since by assumption } (1+a)^k \geq 1+ka\text{]} \\ &= 1+ka+a+ka^2 \\ &= 1+(k+1)a+ka^2 \\ &\geq 1+(k+1)a && \text{[since } ka^2 > 0\text{]}\end{aligned}$$

Hence also true for $n=k+1$.

The statement is true for $n=1$ and if true for $n=k$, then it is also true for $n=k+1$. Hence the statement is true for all positive integers *n* by induction.

28. The square matrices A and B are such that $AB = BA$.
Prove by induction that $A^nB = BA^n$ for all integers $n \geq 1$.

First prove true for $n = 1$, ie prove that $A^1B = BA^1$.

LHS $= A^1B = AB = BA = BA^1 =$ RHS

LHS = RHS, hence true for $n = 1$.

Assume true for $n = k$, ie assume that $A^kB = BA^k$.

Now prove also true for $n = k+1$, ie prove that $A^{k+1}B = BA^{k+1}$.

$A^{k+1}B = (A^kA^1)B$

$= A^k(AB)$

$= A^k(BA)$ [since $AB = BA$]

$= (A^kB)A$

$= (BA^k)A$ [since by assumption $A^kB = BA^k$]

$= B(A^kA^1)$

$= BA^{k+1}$

Hence also true for $n = k+1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k+1$.
Hence the statement is true for all integers $n \geq 1$ by induction.

29.(a) We will use a **direct proof** to prove that the product of two odd integers is odd.

Let a and b be two odd integers.
Then $a = 2k + 1$ and $b = 2p + 1$ for some integers k and p.

$$\begin{aligned} ab &= (2k+1)(2p+1) \\ &= 4kp + 2k + 2p + 1 \\ &= 2(2kp + k + p) + 1 \\ &= 2m + 1 \qquad \text{where } m = 2kp + k + p \end{aligned}$$

$ab = 2m + 1$ for some integer m, hence ab is odd.

(b) Let p be an odd integer.
Use the result of (a) to prove by induction that p^n is odd for all positive integers n.

First prove true for $n = 1$, ie prove that p^1 is odd.

$p^1 = p$ and since p is odd, this means that p^1 is odd, hence true for $n = 1$.

Assume true for $n = k$, ie assume that p^k is odd.

Now prove also true for $n = k + 1$, ie prove that p^{k+1} is odd.

$$p^{k+1} = p^k p^1 = p^k p$$

Now p is odd and by assumption, p^k is odd.
Hence p^{k+1} is the product of two odd integers and the product of two odd integers is odd from (a).
Thus p^{k+1} is odd, hence true for $n = k + 1$.

The statement is true for $n = 1$ and if true for $n = k$, then it is also true for $n = k + 1$.
Hence the statement is true for all positive integers n by induction.