ADVANCED HIGHER MATHEMATICS

Exam Questions on Number Theory and Methods of Proof

1. Use the Euclidean algorithm to obtain the greatest common divisor of 1326 and 14654, expressing it in the form $1326a + 14654b$, where a and b are integers.

2. Use the Euclidean algorithm to find integers a and b such that $306a + 119b = 17$.

3. Use the Euclidean algorithm to find integers a and b such that $1595a + 1218b = 29$.

4. Use the Euclidean algorithm to obtain the greatest common divisor of 1204 and 833, expressing it in the form $1204a + 833b$, where a and b are integers.

5. Use the Euclidean algorithm to find integers p and q such that $599p + 53q = 1$.

6. Use the Euclidean algorithm to find integers x and y such that

$$149x + 139y = 1.$$

7. Use the Euclidean algorithm to find integers p and q such that

$$3066p + 713q = 1.$$

8. Show that the greatest common divisor of 487 and 729 is 1.

Hence find integers x and y such that $487x + 729y = 1$.

9. Use the Euclidean algorithm to show that $(231, 17) = 1$ where (a, b) denotes the highest common factor of a and b.

Hence find integers x and y such that $231x + 17y = 1$.

10. Use the division algorithm to express 1234_{10} in base 7.

11. Prove directly that:

(a)   the sum of any three consecutive integers is divisible by 3;

(b)   any odd integer can be expressed as the sum of two consecutive integers.

12. Prove that the difference between the squares of two consecutive odd numbers is divisible by 8.

13. Let n be a natural number.
For each of the following statements decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A   If n is a multiple of 9 then so is n^2.

B   If n^2 is a multiple of 9 then so is n.

14. For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A   If a positive integer p is prime, then so is $2p+1$.

B   If a positive integer n has remainder 1 when divided by 3, then n^3 also has remainder 1 when divided by 3.

15. For each of the following statements, decide whether it is true or false and prove your conclusion.

A   For all natural numbers m, if m^2 is divisible by 4, then m is divisible by 4.

B   The cube of any odd integer p plus the square of any even integer q is always odd.

16. Given that $p(n) = n^2 + n$, where n is a positive integer, consider the statements:

A $p(n)$ is always even

B $p(n)$ is always a multiple of 3.

For each statement, prove it if it is true or, otherwise, disprove it.

17. For all natural numbers n, prove whether the following statements are true or false.

(a) $n^3 - n$ is always divisible by 6.

(b) $n^3 + n + 5$ is always prime.

18. Prove by contradiction that if x is an irrational number, then $2 + x$ is irrational.

19. (a)   Given that m and n are positive integers state the negation of the statement:
m is even or n is even.

(b)   By considering the contrapositive of the following statement:

if mn is even then m is even or n is even,

prove that the statement is true for all positive integers m and n.

20. Let n be an integer.

Using proof by contrapositive, show that if n^2 is even, then n is even.

21. The matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. Prove by induction that

$$A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix},$$

where n is any positive integer.

22. Given A is the matrix $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$, prove by induction that

$$A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, \quad n \geq 1.$$

23. Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers n.

24. Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 for all integers $n \geq 2$.

25. Prove by induction that $\dfrac{d^n}{dx^n}(xe^x) = (x+n)e^x$ for all integers $n \geq 1$.

26. Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$.

[You may use without proof the following identities: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$]

27. Prove by induction that for $a > 0$,

$$(1+a)^n \geq 1 + na$$

for all positive integers n.

28. The square matrices A and B are such that $AB = BA$.
Prove by induction that $A^n B = BA^n$ for all integers $n \geq 1$.

29. (a) Prove that the product of two odd integers is odd.

(b) Let p be an odd integer.

Use the result of (a) to prove by induction that p^n is odd for all positive integers n.