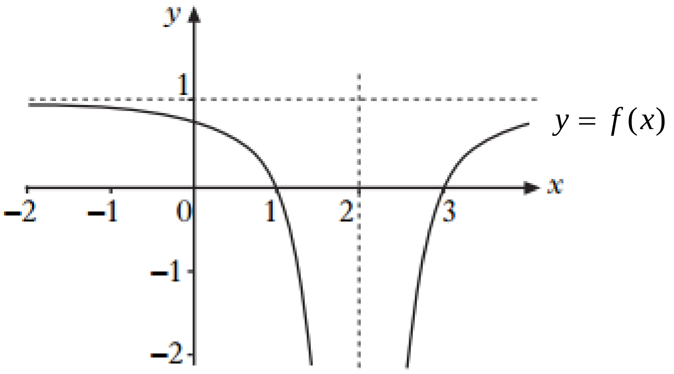
ADVANCED HIGHER MATHEMATICS

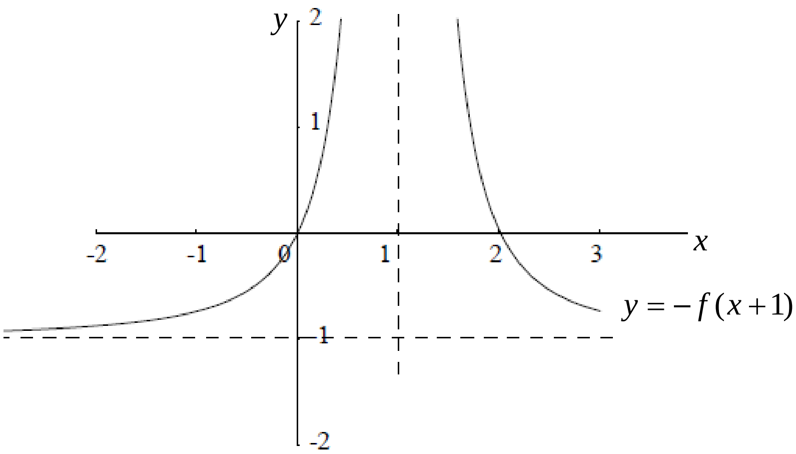
Solutions to Exam Questions on Functions and Graphs

1. The graph of $y = f(x)$ is shown below.



To obtain the graph of $y = -f(x+1)$, you need to move the graph of $y = f(x)$ 1 unit to the left to obtain the graph of $y = f(x+1)$ and then reflect this graph in the *x*-axis.

The graph of $y = -f(x+1)$ is shown below.

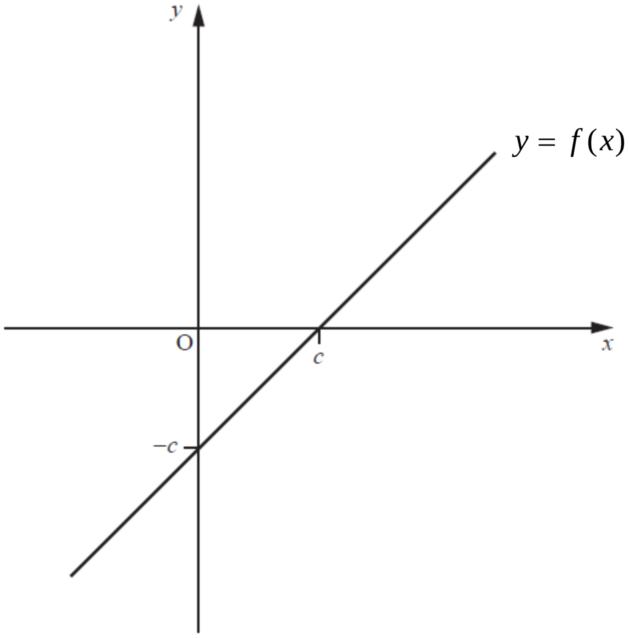


The equation of the vertical asymptote is $x = 1$ and the equation of the non-vertical asymptote is $y = -1$.

Note

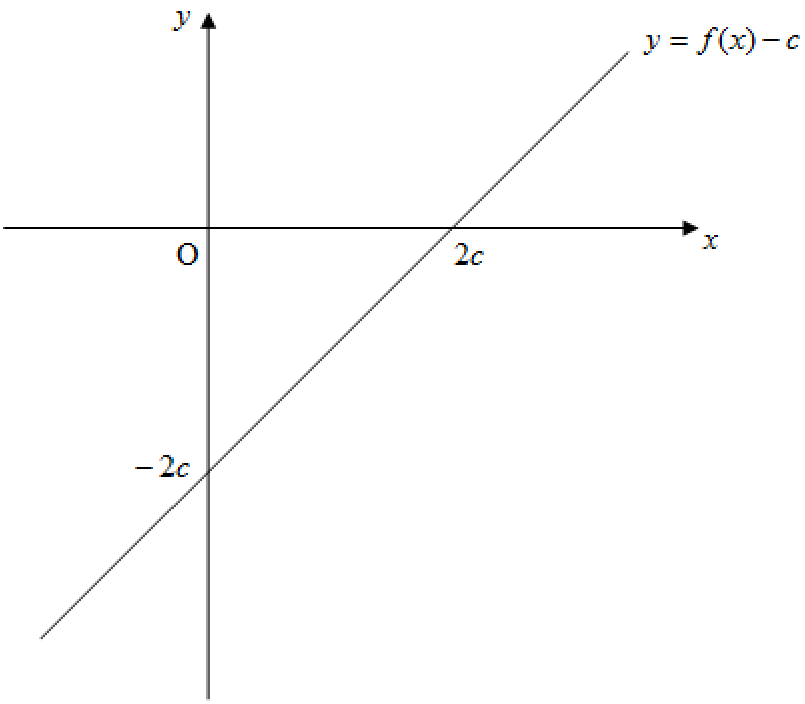
The two steps to obtain the graph of $y = -f(x+1)$ can be carried out in reverse order, ie reflect the graph of $y = f(x)$ in the *x*-axis first to obtain the graph of $y = -f(x)$ and then move this graph 1 unit to the left to obtain the graph of $y = -f(x+1)$.

2. The graph of $y = f(x)$ is shown below.

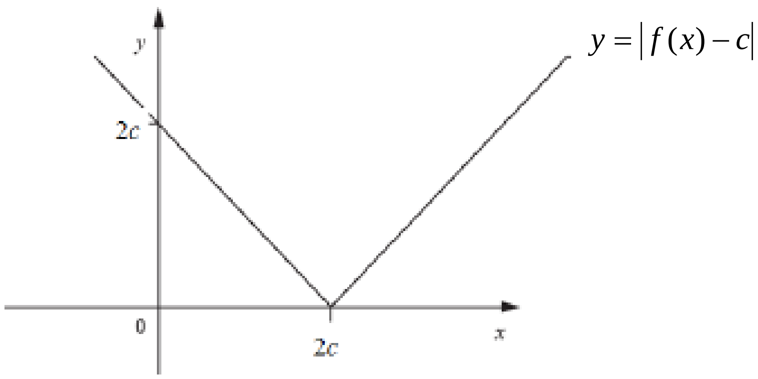


(a) To sketch the graph of $y = |f(x) - c|$, first sketch the graph of $y = f(x) - c$.

To obtain the graph of $y = f(x) - c$, simply move the graph of $y = f(x)$ down c units. The graph of $y = f(x) - c$ is shown below.

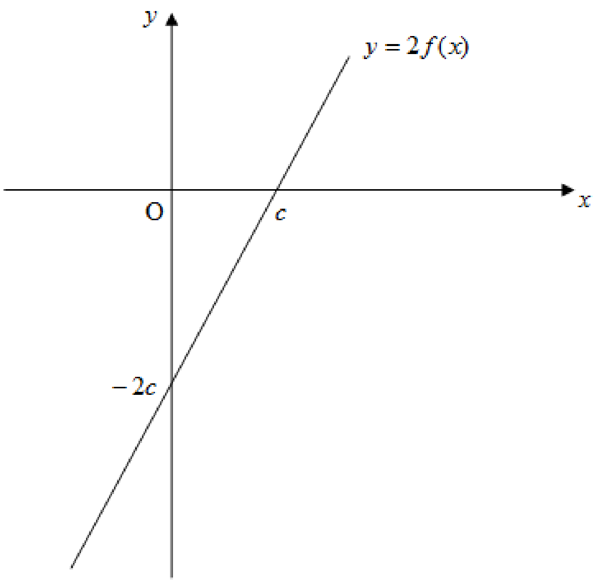


The graph of $y = |f(x) - c|$ is shown below.

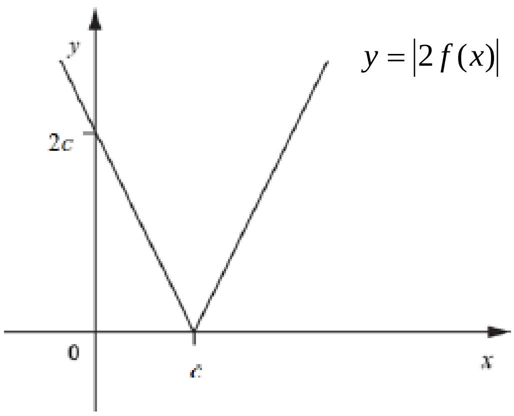


(b) To sketch the graph of $y = |2f(x)|$, first sketch the graph of $y = 2f(x)$.

To obtain the graph of $y = 2f(x)$, multiply all the y-coordinates on the graph of $y = f(x)$ by 2. The graph of $y = 2f(x)$ is shown below.



The graph of $y = |2f(x)|$ is shown below.



3.(a) $y = \dfrac{x}{1+x^2}$

To differentiate y, use the **quotient rule** as y is the quotient of two functions.

$$\frac{dy}{dx} = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

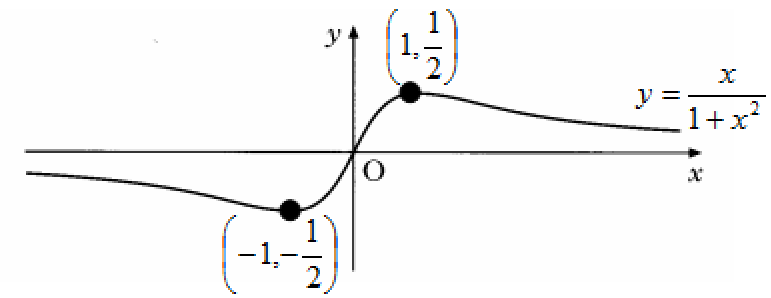
Stationary points occur when $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{1-x^2}{(1+x^2)^2} = 0 \Rightarrow 1 - x^2 = 0$

$$\Rightarrow \quad x^2 = 1$$
$$\Rightarrow \quad x = \pm 1$$

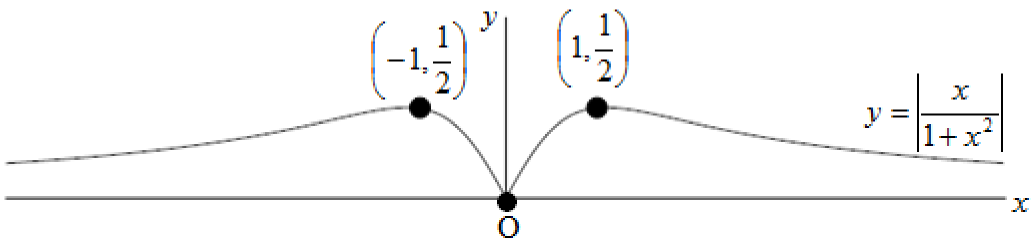
When $x = -1$: $y = \dfrac{-1}{1+(-1)^2} = \dfrac{-1}{2} \Rightarrow \left(-1, -\dfrac{1}{2}\right)$ is a stationary point

When $x = 1$: $y = \dfrac{1}{1+1^2} = \dfrac{1}{2} \Rightarrow \left(1, \dfrac{1}{2}\right)$ is a stationary point

(b) The graph of $y = \dfrac{x}{1+x^2}$ is shown below.



The graph of $y = \left|\dfrac{x}{1+x^2}\right|$ is shown below.



The graph of $y = \left|\dfrac{x}{1+x^2}\right|$ has three critical points at $\left(-1, \dfrac{1}{2}\right)$, (0, 0) and $\left(1, \dfrac{1}{2}\right)$.

4.(a) $y = \dfrac{x^3}{x-2}$

The vertical asymptote occurs when $x - 2 = 0 \quad \Rightarrow \quad x = 2$

The equation of the vertical asymptote is $x = 2$.

(b) To differentiate *y*, use the **quotient rule** as *y* is the quotient of two functions.

$$\frac{dy}{dx} = \frac{(x-2)3x^2 - x^3(1)}{(x-2)^2} = \frac{3x^2(x-2) - x^3}{(x-2)^2} = \frac{3x^3 - 6x^2 - x^3}{(x-2)^2} = \frac{2x^3 - 6x^2}{(x-2)^2}$$

Stationary points occur when $\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{2x^3 - 6x^2}{(x-2)^2} = 0 \quad \Rightarrow \quad 2x^3 - 6x^2 = 0$

$$\Rightarrow \quad 2x^2(x-3) = 0$$

$$\Rightarrow \quad x = 0, \ x = 3$$

When $x = 0$: $\quad y = \dfrac{0^3}{0-2} = 0 \qquad \Rightarrow \quad$ (0, 0) is a stationary point

When $x = 3$: $\quad y = \dfrac{3^3}{3-2} = \dfrac{27}{1} = 27 \quad \Rightarrow \quad$ (3, 27) is a stationary point

(c) The stationary points of the graph of $y = \dfrac{x^3}{x-2}$ are (0, 0) and (3, 27).

The stationary points of the graph of $y = \left|\dfrac{x^3}{x-2}\right|$ are (0, 0) and (3, 27).

The stationary points of the graph of $y = \left|\dfrac{x^3}{x-2}\right| + 1$ are (0, 1) and (3, 28).

5.(a) $f(x)=\dfrac{x^2+6x+12}{x+2}$

Note that $f(x)$ is an **improper algebraic fraction** since the degree of the numerator is greater than or equal to the degree of the denominator, so algebraic long division can be used to divide the numerator by the denominator.

$$
\begin{array}{r|l}
 & \quad\;\; x+4 \\
\hline
x+2 & x^2+6x+12 \\
 & x^2+2x \\
\hline
 & \quad\;\; 4x+12 \\
 & \quad\;\; 4x+\;8 \\
\hline
 & \qquad\quad 4
\end{array}
$$

Hence $f(x)=x+4+\dfrac{4}{x+2}$ and $f(x)=ax+b+\dfrac{b}{x+2}$ where $a=1$ and $b=4$,

(b) $y=x+4+\dfrac{4}{x+2}$

The vertical asymptote occurs when $x+2=0 \quad \Rightarrow \quad x=-2$

The equation of the vertical asymptote is $x=-2$.

The non-vertical asymptote occurs when $x \to \pm\infty$.

The expression $y=x+4+\dfrac{4}{x+2}$ shows that as $x \to \pm\infty$, $y \to x+4$ since $\dfrac{4}{x+2} \to 0$.

Hence the equation of the non-vertical asymptote is $y=x+4$.

(c) The most efficient way to find the stationary points is to differentiate $y = x + 4 + \frac{4}{x+2}$.

$$y = x + 4 + \frac{4}{x+2} \quad \Rightarrow \quad y = x + 4 + 4(x+2)^{-1}$$

$$\frac{dy}{dx} = 1 - 4(x+2)^{-2} \times 1 = 1 - 4(x+2)^{-2} = 1 - \frac{4}{(x+2)^2}$$

Stationary points occur when $\frac{dy}{dx} = 0 \quad \Rightarrow \quad 1 - \frac{4}{(x+2)^2} = 0$

$$\Rightarrow \quad 1 = \frac{4}{(x+2)^2}$$

$$\Rightarrow \quad (x+2)^2 = 4$$

$$\Rightarrow \quad x + 2 = 2 \quad \text{or} \quad x + 2 = -2$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = -4$$

When $x = -4$: $\quad y = -4 + 4 + \frac{4}{-4+2} = -2 \quad \Rightarrow$ (–4, –2) is a stationary point

When $x = 0$: $\quad y = 0 + 4 + \frac{4}{0+2} = 6 \quad \Rightarrow$ (0, 6) is a stationary point

To determine the nature of each stationary point, use the second derivative test.

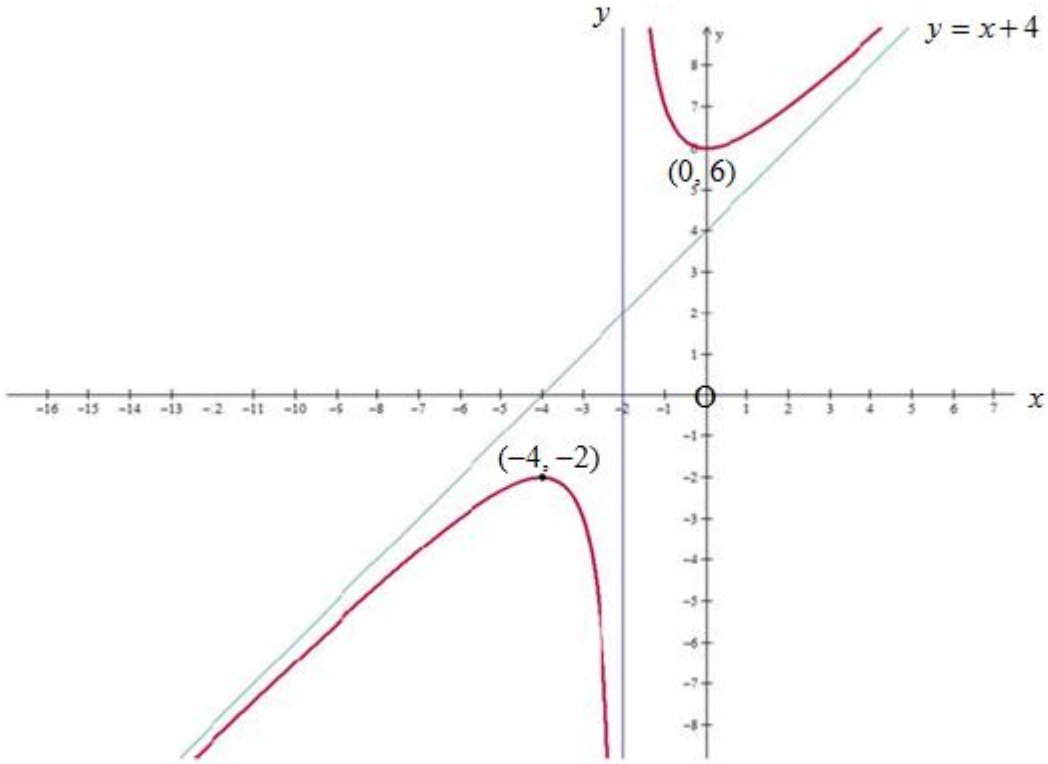
$$\frac{dy}{dx} = 1 - 4(x+2)^{-2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 8(x+2)^{-3} \times 1 = \frac{8}{(x+2)^3}$$

When $x = -4$: $\quad \frac{d^2y}{dx^2} = \frac{8}{(-4+2)^3} = \frac{8}{(-2)^3} = \frac{8}{-8} = -1 < 0 \quad \Rightarrow$ (–4, –2) is a max tp

When $x = 0$: $\quad \frac{d^2y}{dx^2} = \frac{8}{(0+2)^3} = \frac{8}{2^3} = \frac{8}{8} = 1 > 0 \quad \Rightarrow$ (0, 6) is a min tp

(–4, –2) is a maximum turning point and (0, 6) is a minimum turning point.

(d)



(e) The graph shows that equation $f(x) = k$ has no real solutions when $-2 < k < 6$.

6.(a) (i) $y = \dfrac{x^2 + 3x + 6}{x + 2}$

The vertical asymptote occurs when $x + 2 = 0 \quad \Rightarrow \quad x = -2$

The equation of the vertical asymptote is $x = -2$.

(ii) Note that $f(x)$ is an **improper algebraic fraction** since the degree of the numerator is greater than or equal to the degree of the denominator, so algebraic long division can be used to divide the numerator by the denominator.

$$
\begin{array}{r|l}
 & x + 1 \\ \hline
x + 2 & x^2 + 3x + 6 \\
 & x^2 + 2x \\ \hline
 & x + 6 \\
 & x + 2 \\ \hline
 & 4
\end{array}
$$

Hence $y = x + 1 + \dfrac{4}{x + 2}$.

The non-vertical asymptote occurs when $x \to \pm\infty$.

The expression $y = x + 1 + \dfrac{4}{x + 2}$ shows that as $x \to \pm\infty$, $y \to x + 1$ since $\dfrac{4}{x + 2} \to 0$.

Hence the equation of the non-vertical asymptote is $y = x + 1$.

(b) The most efficient way to find the stationary points is to differentiate $y = x + 1 + \dfrac{4}{x+2}$.

$$y = x + 1 + \frac{4}{x+2} \quad \Rightarrow \quad y = x + 1 + 4(x+2)^{-1}$$

$$\frac{dy}{dx} = 1 - 4(x+2)^{-2} \times 1 = 1 - 4(x+2)^{-2} = 1 - \frac{4}{(x+2)^2}$$

Stationary points occur when $\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad 1 - \dfrac{4}{(x+2)^2} = 0$

$$\Rightarrow \quad 1 = \frac{4}{(x+2)^2}$$

$$\Rightarrow \quad (x+2)^2 = 4$$

$$\Rightarrow \quad x + 2 = 2 \quad \text{or} \quad x + 2 = -2$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = -4$$

When $x = -4$: $\quad y = -4 + 1 + \dfrac{4}{-4+2} = -5 \quad \Rightarrow \quad (-4, -5)$ is a stationary point

When $x = 0$: $\quad y = 0 + 1 + \dfrac{4}{0+2} = 3 \quad \Rightarrow \quad (0, 3)$ is a stationary point

To determine the nature of each stationary point, use the second derivative test.

$$\frac{dy}{dx} = 1 - 4(x+2)^{-2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 8(x+2)^{-3} \times 1 = \frac{8}{(x+2)^3}$$

When $x = -4$: $\quad \dfrac{d^2y}{dx^2} = \dfrac{8}{(-4+2)^3} = \dfrac{8}{(-2)^3} = \dfrac{8}{-8} = -1 < 0 \quad \Rightarrow \quad (-4, -5)$ is a max tp

When $x = 0$: $\quad \dfrac{d^2y}{dx^2} = \dfrac{8}{(0+2)^3} = \dfrac{8}{2^3} = \dfrac{8}{8} = 1 > 0 \quad \Rightarrow \quad (0, 3)$ is a min tp

$(-4, -5)$ is a maximum turning point and $(0, 3)$ is a minimum turning point.

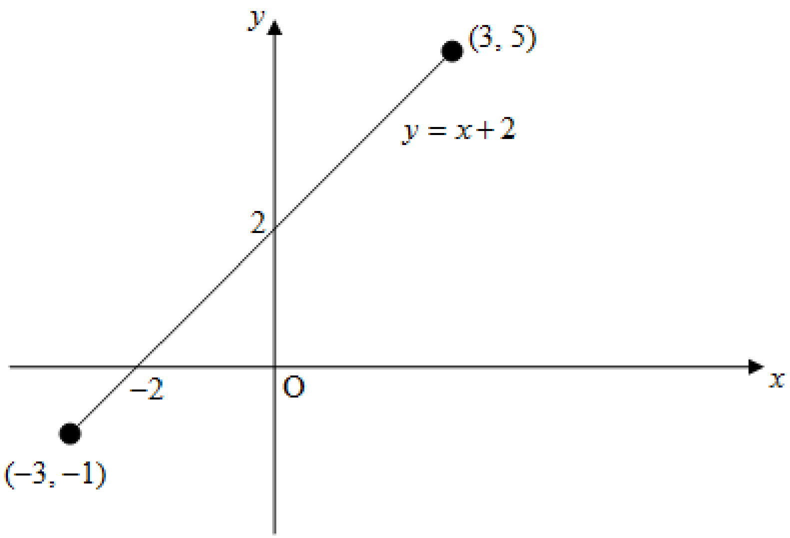
7.(a) $f(x) = |x+2|$

To sketch the graph of $y = f(x)$ for $-3 \le x \le 3$, first sketch the graph of $y = x+2$ for $-3 \le x \le 3$.

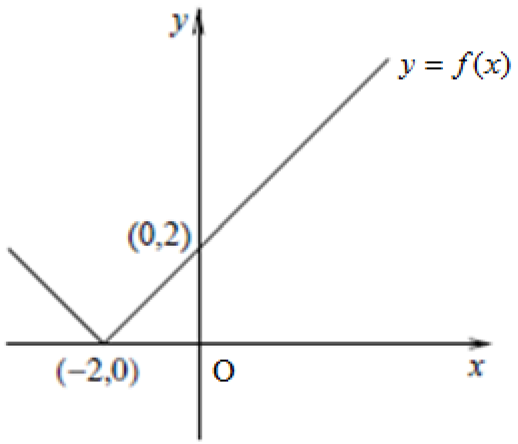
Using the formula $y = x+2$ to calculate the y-coordinates, the graph of $y = x+2$ is a straight line passing through the following points:

(−3, −1), (−2, 0), (−1, 1), (0, 2), (1, 3), (2, 4), (3, 5)

A sketch of the graph of $y = x+2$ for $-3 \le x \le 3$ is shown below.



The graph of $y = |x+2|$ for $-3 \le x \le 3$ is shown below.

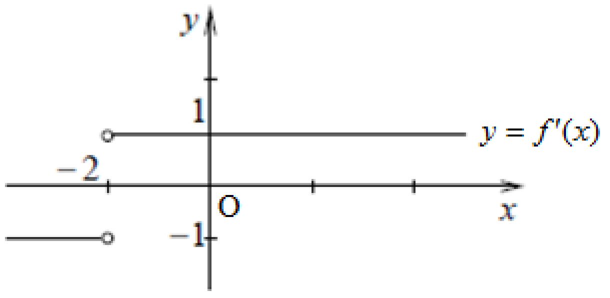


(b) To sketch the graph of $y = f'(x)$, remember that $f'(x)$ is the **gradient** of the graph of $y = f(x)$.

The straight line with equation $y = x + 2$ has gradient 1.

Hence $f'(x) = -1$ for $x < -2$ and $f'(x) = 1$ for $x > -2$.

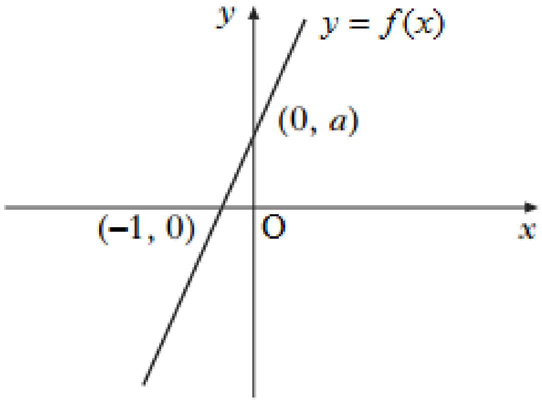
The graph of $y = f'(x)$ is shown below.



<u>Note</u>

$f(x)$ is not differentiable at $x = -2$, so $f'(-2)$ is undefined.

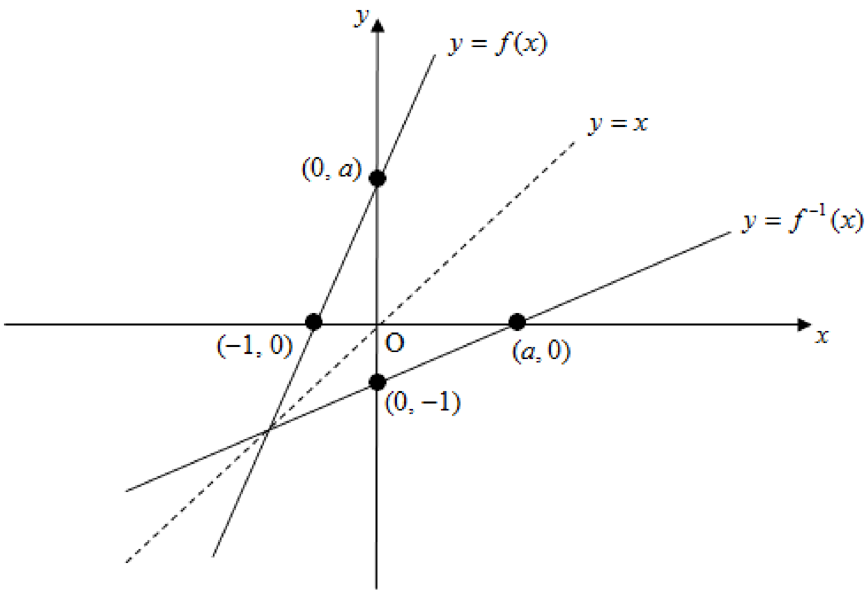
8. The graph of $y = f(x)$ is shown below.



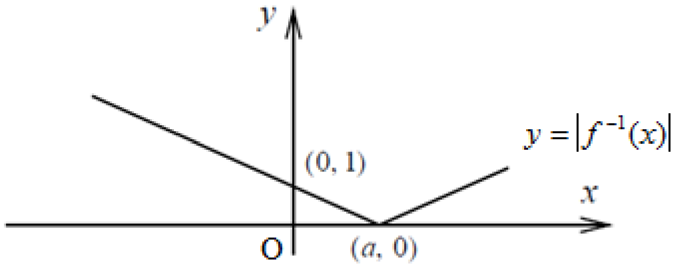
To sketch the graph of $y = \left|f^{-1}(x)\right|$, first sketch the graph of $y = f^{-1}(x)$.

Remember that the graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ in the line with equation $y = x$ and the *x*-coordinates and *y*-coordinates of points swap over. Thus the graph of $y = f^{-1}(x)$ will be a straight line passing through the points (a, 0) and (0, −1).

The graph of $y = f^{-1}(x)$ is shown in the diagram below.



The graph of $y = \left| f^{-1}(x) \right|$ is shown below.



9.(a) $y = \dfrac{x-3}{x+2}$

The vertical asymptote occurs when $x + 2 = 0 \quad \Rightarrow \quad x = -2$

The equation of the vertical asymptote is $x = -2$.

Note that *y* is an **improper algebraic fraction** since the degree of the numerator is equal to the degree of the denominator, so algebraic long division can be used to divide the numerator by the denominator.

$$\begin{array}{r|l} & \quad 1 \\ \hline x+2 & x-3 \\ & x+2 \\ \hline & \;-5 \end{array}$$

Hence $y = 1 + \dfrac{-5}{x+2}$ *or* $y = 1 - \dfrac{5}{x+2}$.

The non-vertical asymptote occurs when $x \to \pm\infty$.

The expression $y = 1 - \dfrac{5}{x+2}$ shows that as $x \to \pm\infty$, $y \to 1$ since $\dfrac{5}{x+2} \to 0$.

Hence the equation of the non-vertical asymptote is $y = 1$.

(b) Differentiate $y = 1 - \dfrac{5}{x+2} \quad \Rightarrow \quad y = 1 - 5(x+2)^{-1}$

$$\frac{dy}{dx} = 5(x+2)^{-2} \times 1 = 5(x+2)^{-2} = \frac{5}{(x+2)^2}$$

For all *x* where $x \neq -2$, $(x+2)^2 > 0$ and $\dfrac{dy}{dx} > 0$.

Hence for all *x* where $x \neq -2$, $\dfrac{dy}{dx} \neq 0$ and the graph of *f* has no stationary points.

Differentiate $\frac{dy}{dx} = 5(x+2)^{-2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -10(x+2)^{-3} \times 1 = \frac{-10}{(x+2)^3}$

If $x < -2$, $x + 2 < 0$ and $(x+2)^3 < 0$ meaning that $\frac{d^2y}{dx^2} > 0$.

If $x > -2$, $x + 2 > 0$ and $(x+2)^3 > 0$ meaning that $\frac{d^2y}{dx^2} < 0$.

Hence for all *x* where $x \neq -2$, $\frac{d^2y}{dx^2} \neq 0$ and the graph of *f* has no points of inflexion.

10. $f(x) = x^2 \sin x$

To determine whether the function *f* is odd, even or neither, find an expression for $f(-x)$ in its simplest form and compare this expression to $f(x)$.

$$\begin{aligned} f(-x) &= (-x)^2 \sin(-x) \\ &= x^2(-\sin x) \qquad [\text{since } (-x)^2 = x^2 \text{ and } \sin(-x) = -\sin x\,] \\ &= -x^2 \sin x \\ &= -f(x) \end{aligned}$$

$f(-x) = -f(x)$ for all *x*, hence *f* is an **odd** function.

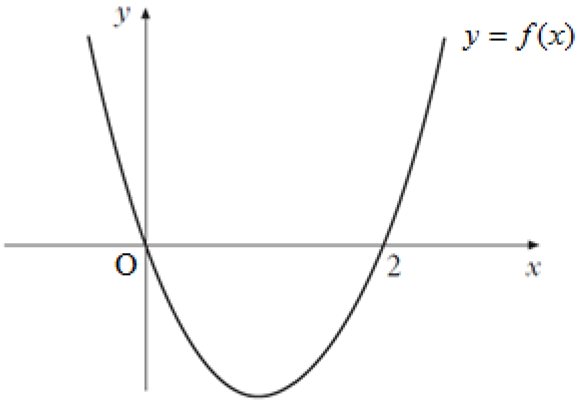
11. $f(x) = x^4 \sin 2x$

To determine whether the function *f* is odd, even or neither, find an expression for $f(-x)$ in its simplest form and compare this expression to $f(x)$.

$$\begin{aligned} f(-x) &= (-x)^4 \sin\big(2(-x)\big) \\ &= (-x)^4 \sin(-2x) \\ &= x^4(-\sin 2x) \qquad [\text{since } (-x)^4 = x^4 \text{ and } \sin(-2x) = -\sin 2x\,] \\ &= -x^4 \sin 2x \\ &= -f(x) \end{aligned}$$

$f(-x) = -f(x)$ for all *x*, hence *f* is an **odd** function.

12. The graph of $y = f(x)$ is shown below.



There are two ways of investigate whether the function f is odd, even or neither.

Method 1

This method involves knowledge of the properties of the graphs of odd and even functions. Remember that the graph of an even function will always be symmetrical about the *y*-axis and the graph of an odd function will always have half-turn rotational symmetry about the origin.

The graph of *f* is not symmetrical about the y-axis, so *f* is not an even function.

The graph of *f* does not have half-turn rotational symmetry about the origin, so *f* is not an odd function.

Hence *f* is **neither** odd nor even.

Method 2

This method involves developing and using a formula for $f(x)$.

f is a quadratic function with roots at $x = 0$ and $x = 2$.

Hence $f(x) = kx(x-2)$ for some constant *k*.

$f(x) = kx^2 - 2kx$ for some constant *k*.

To determine whether the function *f* is odd, even or neither, find an expression for $f(-x)$ in its simplest form and compare this expression to $f(x)$.

$f(x) = kx^2 - 2kx$

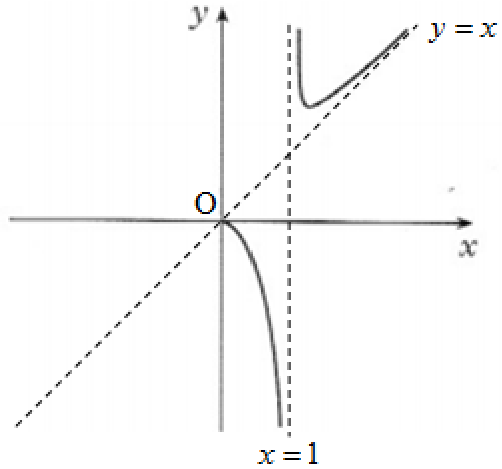
$$f(-x) = k(-x)^2 - 2k(-x)$$
$$= kx^2 + 2kx \qquad \text{[since } (-x)^2 = x^2 \text{]}$$

We can see from this expression that $f(-x) \neq f(x)$, so *f* is not an even function.

We can also see from this expression that $f(-x) \neq -f(x)$ since $-f(x) = -kx^2 + 2kx$, so *f* is not an odd function.

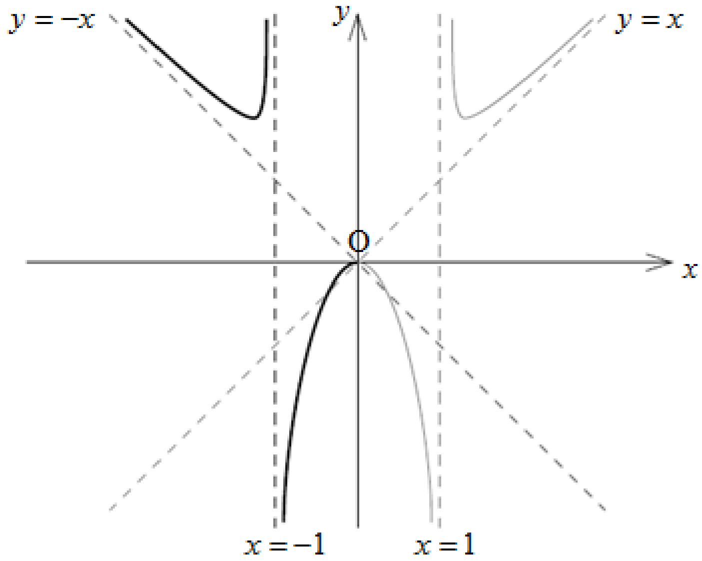
Hence *f* is **neither** odd nor even.

13. The graph of part of the function *f* with two of its asymptotes is shown below.



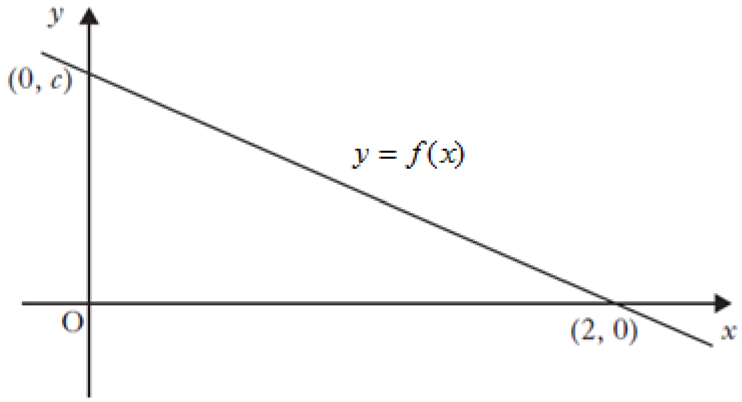
We are given that *f* is an even function, so the graph of *f* will be symmetrical about the *y*-axis.

The complete graph of the function *f* is shown below.



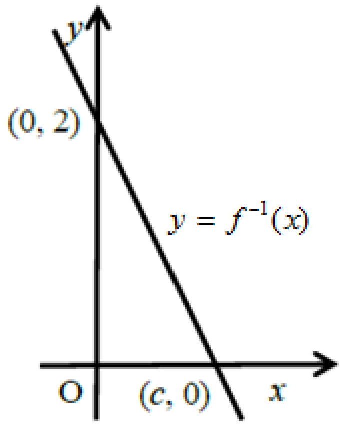
The equations of the other two asymptotes are $y=-x$ and $x=-1$.

14.(a) The graph of $y = f(x)$ is shown below.



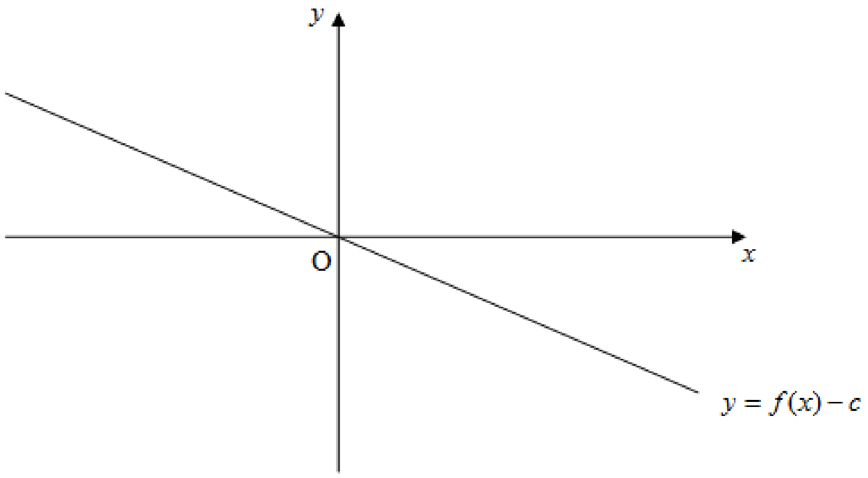
Remember that the graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ in the line with equation $y = x$ and the x-coordinates and y-coordinates of points swap over. Thus the graph of $y = f^{-1}(x)$ will be a straight line passing through the points (c, 0) and (0, 2).

The graph of $y = f^{-1}(x)$ is shown below.



(b) $f(x)+k$ will be an odd function when the graph of $y = f(x)+k$ has half-turn rotational symmetry about the origin.

Consider the graph of $y = f(x)-c$.
This graph is obtained by moving the graph of $y = f(x)$ down c units and is shown below.

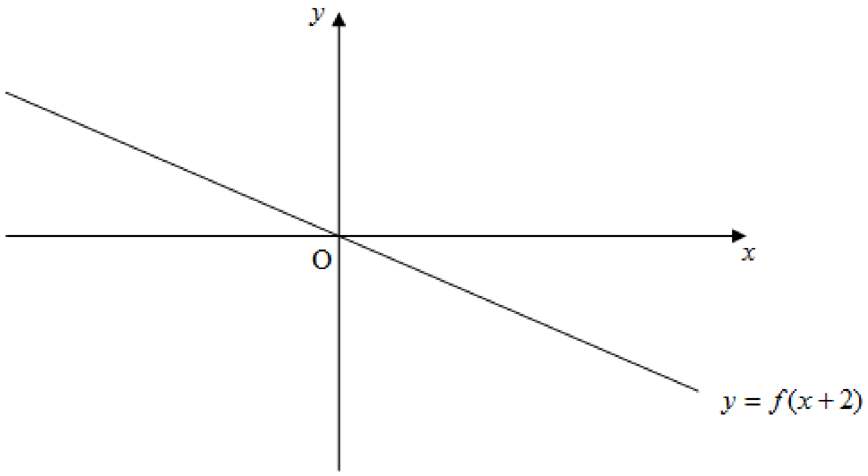


The graph of $y = f(x)-c$ passes through O and clearly has half-turn rotational symmetry about the origin, so $f(x)-c$ is an odd function.

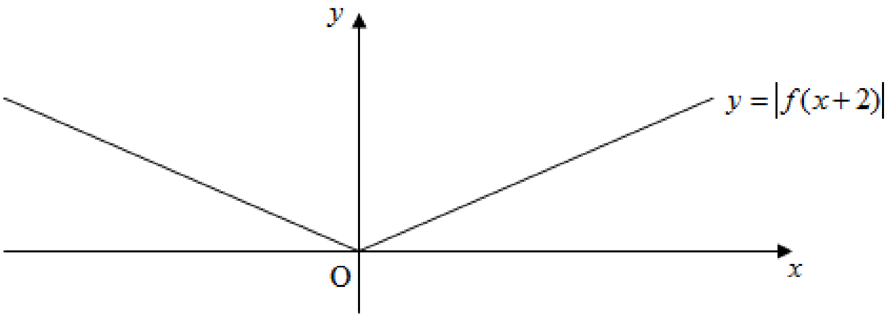
Hence $f(x)+k$ is an odd function when $k = -c$.

(c) $|f(x+h)|$ will be an even function when the graph of $y = |f(x+h)|$ is symmetrical about the y-axis.

Consider the graph of $y = f(x+2)$.
This graph is obtained by moving the graph of $y = f(x)$ 2 units to the left and is shown below.



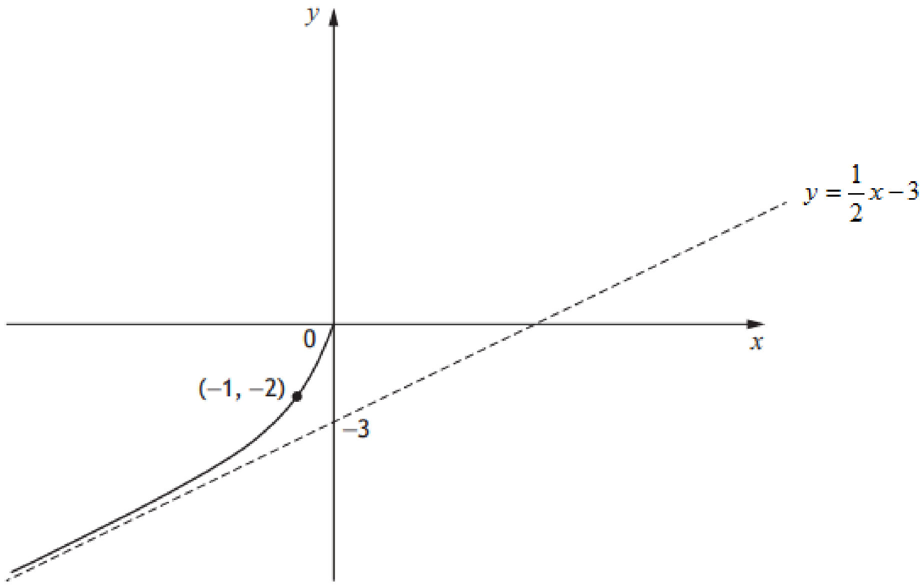
The graph of $y = |f(x+2)|$ is shown below.



The graph of $y = |f(x+2)|$ is symmetrical about the *y*-axis, so $|f(x+2)|$ is an odd function.

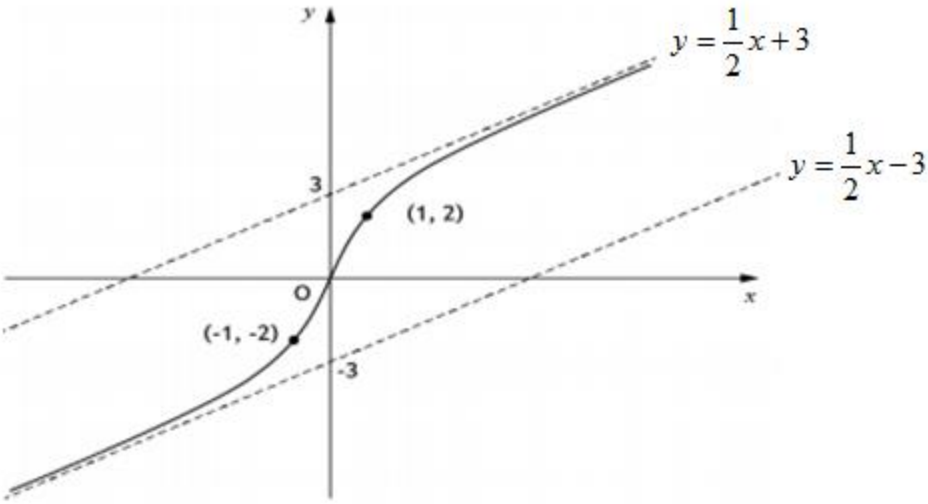
Hence $|f(x+h)|$ is an odd function when $h = 2$.

15.(a) The graph of part of the function *f* is shown below.



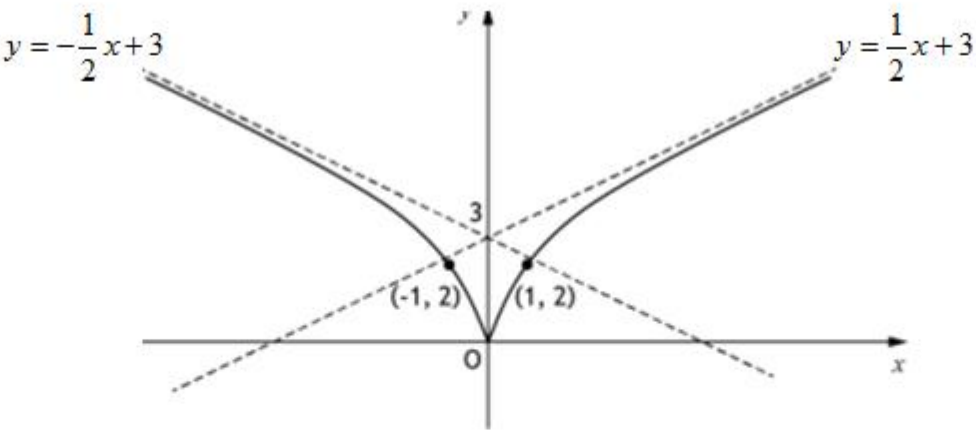
We are given that f is an odd function, so the graph of f will have half-turn rotational symmetry about the origin.

The complete graph of the function *f* is shown below.

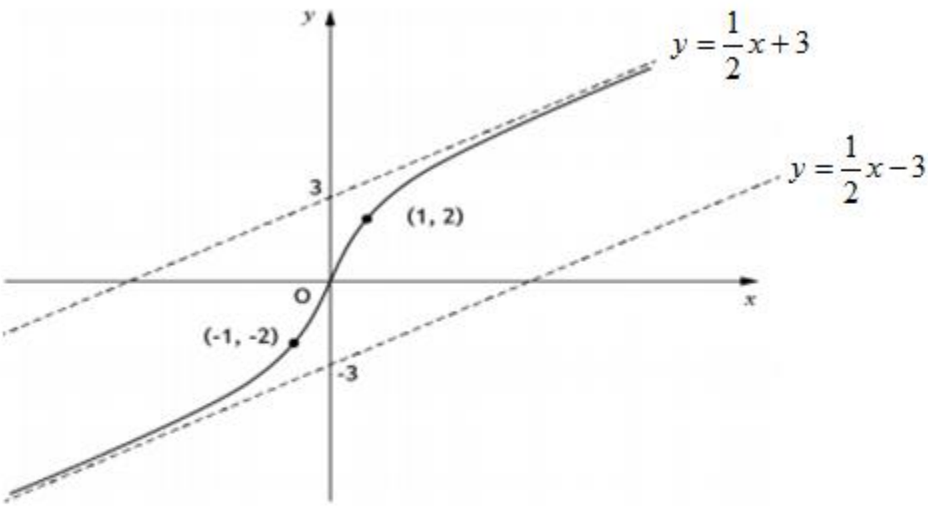


(b) $g(x) = |f(x)|$

The graph of g is shown below.



(c) To determine the range of values of $f'(x)$, consider the gradient of the graph of $f(x)$ since $f'(x)$ is the gradient of the graph of $f(x)$.
The graph of $f(x)$ is shown again below.



At the start of the graph, $f(x)$ is very close to the asymptote $y = \frac{1}{2}x - 3$ which has gradient $\frac{1}{2}$. At the start, the gradient of the graph is slightly greater than $\frac{1}{2}$, hence $\frac{1}{2} < f'(x)$.

Following the graph to the origin, we can see that the gradient of the graph steadily increases and at the origin the gradient of the graph is 2 since we are given that $f'(0) = 2$.

Hence $\frac{1}{2} < f'(x) \leq 2$ for $x < 0$.

After the graph passes through the origin, the gradient of the graph steadily decreases and the gradient tends towards $\frac{1}{2}$ at the end of the graph as the graph approaches the asymptote $y = \frac{1}{2}x + 3$.

Hence the range of values of $f'(x)$ is given by $\frac{1}{2} < f'(x) \leq 2$.

<u>Note</u>

The gradient of the graph never actually equals $\frac{1}{2}$ but the gradient of the graph does equal 2 at the origin. Care must be taken to use precise inequality symbols when expressing the range of values of $f'(x)$.

16.(a) $g(x) = f(x) + f(-x)$

To determine whether the function $g(x)$ is odd, even or neither, find an expression for $g(-x)$ in its simplest form and compare this expression to $g(x)$.

$$
\begin{aligned}
g(-x) &= f(-x) + f\big(-(-x)\big) \\
&= f(-x) + f(x) \\
&= f(x) + f(-x) \\
&= g(x)
\end{aligned}
$$

$g(-x) = g(x)$ for all x, hence $g(x)$ is an even function.

$h(x) = f(x) - f(-x)$

To determine whether the function $h(x)$ is odd, even or neither, find an expression for $h(-x)$ in its simplest form and compare this expression to $h(x)$.

$$
\begin{aligned}
h(-x) &= f(-x) - f\big(-(-x)\big) \\
&= f(-x) - f(x) \\
&= -f(x) + f(-x) \\
&= -\big(f(x) - f(-x)\big) \\
&= -h(x)
\end{aligned}
$$

$h(-x) = -h(x)$ for all x, hence $h(x)$ is an odd function.

(b) $f(x)$ can be expressed in terms of $g(x)$ and $h(x)$ as follows.

$$
\begin{aligned}
g(x) + h(x) &= \big(f(x) + f(-x)\big) + \big(f(x) - f(-x)\big) \\
&= f(x) + f(-x) + f(x) - f(-x) \\
&= 2f(x)
\end{aligned}
$$

Hence $2f(x) = g(x) + h(x) \quad \Rightarrow \quad f(x) = \frac{1}{2}\big(g(x) + h(x)\big)$

$$\Rightarrow \quad f(x) = \frac{1}{2}g(x) + \frac{1}{2}h(x)$$

We know that $g(x)$ is an even function, so $\frac{1}{2}g(x)$ will also be an even function.

We know that $h(x)$ is an odd function, so $\frac{1}{2}h(x)$ will also be an odd function.

Hence $f(x)$ can be expressed as the sum of an even and an odd function.