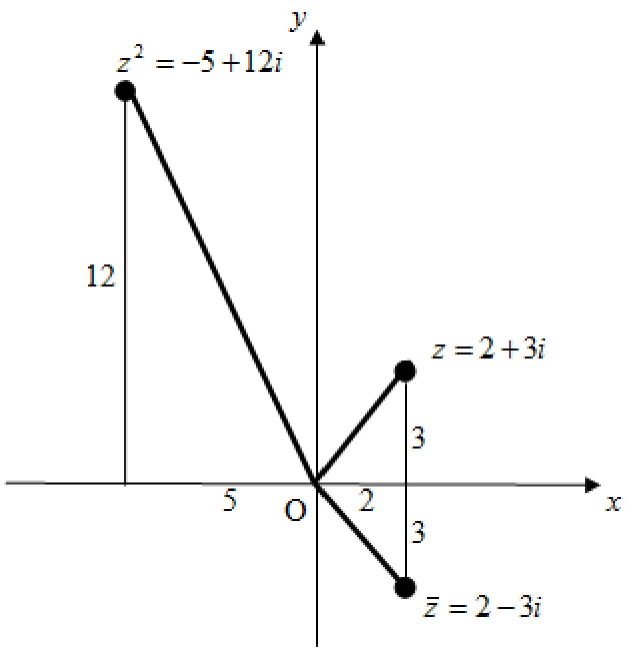
# Solutions to Exam Questions on Complex Numbers 2

**1.** $z = 2 + 3i \quad \rightarrow \quad$ plot (2, 3)

$\bar{z} = 2 - 3i \quad \rightarrow \quad$ plot (2, –3)

$z^2 = (2+3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i \quad \rightarrow \quad$ plot (–5, 12)



**2.** $z = \dfrac{(1+2i)^2}{7-i} = \dfrac{1+4i+4i^2}{7-i} = \dfrac{1+4i-4}{7-i} = \dfrac{-3+4i}{7-i}$

$$= \frac{(-3+4i)(7+i)}{(7-i)(7+i)}$$

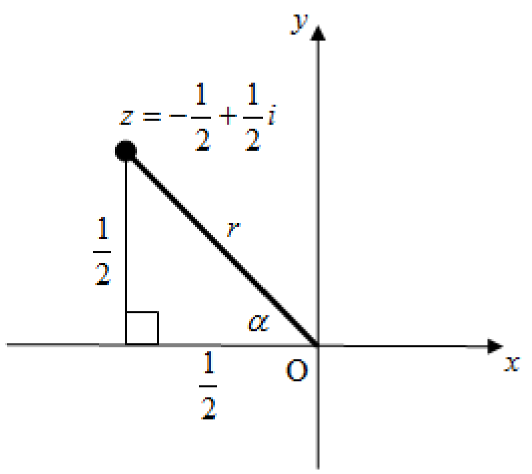
$$= \frac{-21+25i+4i^2}{49-i^2}$$

$$= \frac{-21+25i-4}{49+1}$$

$$= \frac{-25+25i}{50}$$

$$= -\frac{1}{2}+\frac{1}{2}i$$

$z = -\dfrac{1}{2}+\dfrac{1}{2}i \quad \to \quad \text{plot } \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$




$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\tan\alpha = \dfrac{1/2}{1/2} = 1 \quad \Rightarrow \quad \alpha = \tan^{-1}1 = 45^\circ \quad \Rightarrow \quad \theta = 180^\circ - 45^\circ = 135^\circ$

Hence $|z| = \dfrac{1}{\sqrt{2}}$ and $\arg(z) = 135^\circ$.

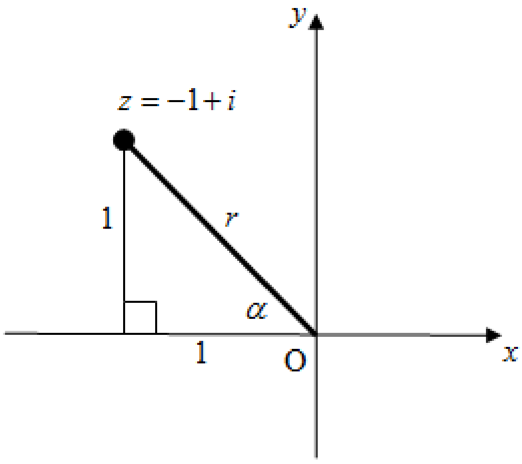
**<u>Note</u>**

The argument of a complex number can be expressed in degrees or radians.

In radians, $\arg(z) = \dfrac{3\pi}{4}$.

**3.** $$\frac{1+3i}{1-2i} = \frac{(1+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{1+5i+6i^2}{1-4i^2} = \frac{1+5i-6}{1+4} = \frac{-5+5i}{5} = -1+i$$

Let $z = \dfrac{1+3i}{1-2i} = -1+i \quad \rightarrow \quad$ plot $(-1, 1)$




$$r = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\tan\alpha = \frac{1}{1} = 1 \quad \Rightarrow \quad \alpha = \tan^{-1} 1 = 45° \quad \Rightarrow \quad \theta = 180° - 45° = 135°$$

Hence $|z| = \sqrt{2}$ and $\arg(z) = 135°$.

**<u>Note</u>**

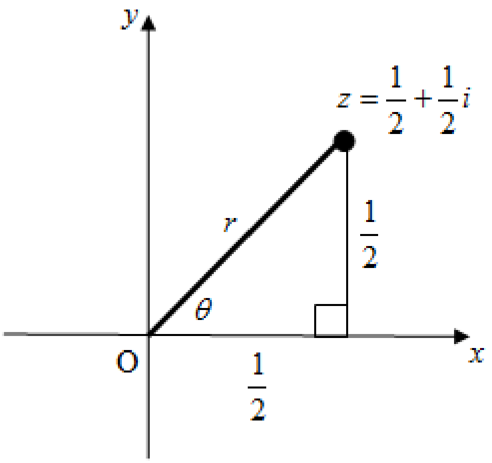
The argument of a complex number can be expressed in degrees or radians.

In radians, $\arg(z) = \dfrac{3\pi}{4}$.

**4.** To determine the modulus and argument of the complex number $\frac{1}{1-i}$, first express $\frac{1}{1-i}$ in the form $a+bi$.

$$\frac{1}{1-i}=\frac{1(1+i)}{(1-i)(1+i)}=\frac{1+i}{1-i^2}=\frac{1+i}{1+1}=\frac{1+i}{2}=\frac{1}{2}+\frac{1}{2}i$$

Let $z=\frac{1}{1-i}=\frac{1}{2}+\frac{1}{2}i \quad \rightarrow \quad$ plot $\left(\frac{1}{2},\frac{1}{2}\right)$




$$r=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{1}{4}}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

$$\tan\theta=\frac{1/2}{1/2}=1 \quad \Rightarrow \quad \theta=\tan^{-1}1=45°$$

Hence $|z|=\frac{1}{\sqrt{2}}$ and $\arg(z)=45°$.

**<u>Note</u>**

The argument of a complex number can be expressed in degrees or radians.

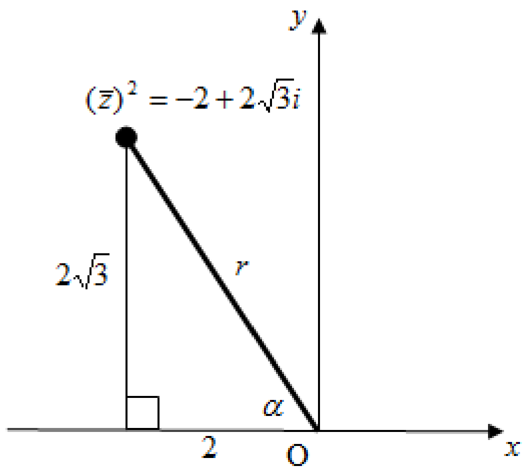
In radians, $\arg(z)=\frac{\pi}{4}$.

**5.** $z = 1 - \sqrt{3}i \quad \Rightarrow \quad \bar{z} = 1 + \sqrt{3}i$

$$(\bar{z})^2 = \left(1 + \sqrt{3}i\right)^2 = 1 + 2\sqrt{3}i + 3i^2 = 1 + 2\sqrt{3}i - 3 = -2 + 2\sqrt{3}i$$

To express $(\bar{z})^2$ in polar form, plot $(\bar{z})^2$ on an Argand diagram and find the modulus and argument of $(\bar{z})^2$.

$$(\bar{z})^2 = -2 + 2\sqrt{3}i \quad \rightarrow \quad \text{plot } \left(-2, 2\sqrt{3}\right)$$




$$r = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\tan\alpha = \frac{2\sqrt{3}}{2} = \sqrt{3} \quad \Rightarrow \quad \alpha = \tan^{-1}\sqrt{3} = 60° \quad \Rightarrow \quad \theta = 180° - 60° = 120°$$

polar form: $(\bar{z})^2 = r(\cos\theta + i\sin\theta) \quad \Rightarrow \quad (\bar{z})^2 = 4(\cos 120° + i\sin 120°)$

**<u>Note</u>**

The argument of a complex number can be expressed in degrees or radians.

In radians, $\theta = \dfrac{2\pi}{3}$ leading to the polar form $(\bar{z})^2 = 4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$.

**6.** $z = -i + \dfrac{1}{1-i}$

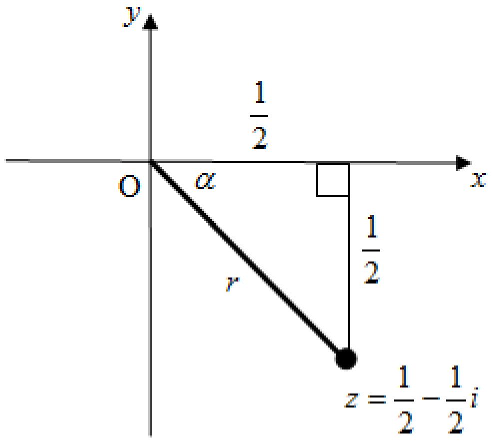
To express z in the form $x + yi$, first simplify $\dfrac{1}{1-i}$.

$$\frac{1}{1-i} = \frac{1(1+i)}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1+i}{1+1} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

Then $z = -i + \dfrac{1}{1-i} = -i + \dfrac{1}{2} + \dfrac{1}{2}i = \dfrac{1}{2} - \dfrac{1}{2}i$.

Hence $z = x + yi$ where $x = \dfrac{1}{2}$ and $y = -\dfrac{1}{2}$.

$z = \dfrac{1}{2} - \dfrac{1}{2}i \quad \rightarrow \quad$ plot $\left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$



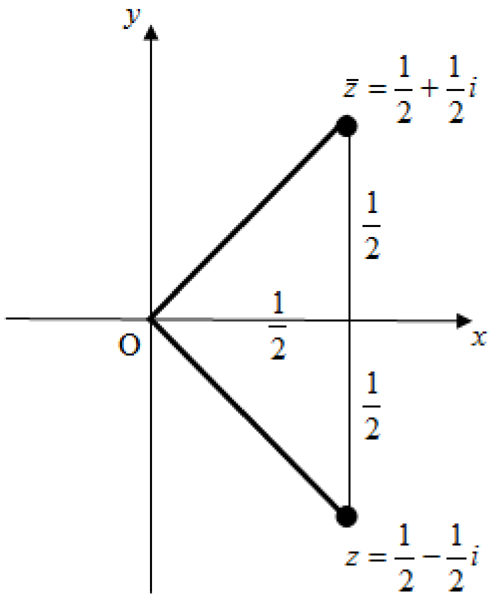

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\tan\alpha = \frac{1/2}{1/2} = 1 \quad \Rightarrow \quad \alpha = \tan^{-1} 1 = 45° \quad \Rightarrow \quad \theta = -45°$$

Hence $|z| = \dfrac{1}{\sqrt{2}}$ and $\arg(z) = -45°$.

$$z = \frac{1}{2} - \frac{1}{2}i \quad \Rightarrow \quad \bar{z} = \frac{1}{2} + \frac{1}{2}i \quad \rightarrow \quad \text{plot} \left( \frac{1}{2}, \frac{1}{2} \right)$$

The complex numbers z and $\bar{z}$ are plotted on the Argand diagram below.



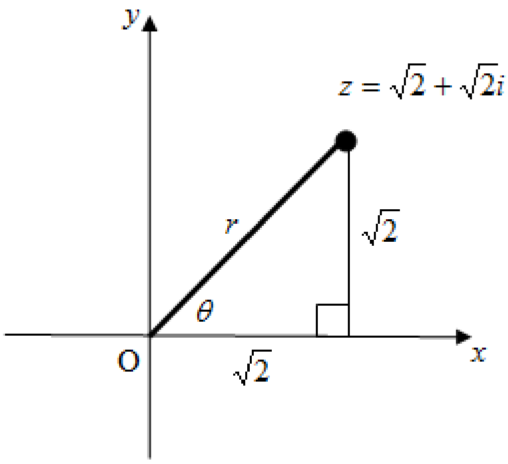

**<u>Notes</u>**

(1) Remember that $-180° < \arg(z) \leq 180°$ or $-\pi < \arg(z) \leq \pi$ .

(2) The argument of a complex number can be expressed in degrees or radians.

In radians, $\arg(z) = -\frac{\pi}{4}$ .

**7.** $z = \sqrt{2}(1+i) = \sqrt{2} + \sqrt{2}i \quad \rightarrow \quad \text{plot } \left(\sqrt{2}, \sqrt{2}\right)$




$$r = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = \sqrt{2+2} = \sqrt{4} = 2$$

$$\tan\theta = \frac{\sqrt{2}}{\sqrt{2}} = 1 \quad \Rightarrow \quad \theta = \tan^{-1} 1 = 45°$$

polar form: $z = r(\cos\theta + i\sin\theta) \quad \Rightarrow \quad z = 2(\cos 45° + i\sin 45°)$

There are several different methods for finding $z^4$.

**Method 1**

By de Moivre's theorem:
$$\begin{aligned} z^4 &= [2(\cos 45° + i\sin 45°)]^4 \\ &= 2^4\left(\cos 4(45°) + i\sin 4(45°)\right) \\ &= 16(\cos 180° + i\sin 180°) \\ &= 16(-1+0i) \\ &= 16(-1) \\ &= -16 \end{aligned}$$

Hence $z^4 + 16 = -16 + 16 = 0$, as required.

**Method 2**

$z^2 = \left(\sqrt{2}(1+i)\right)^2 = 2(1+i)^2 = 2(1+2i+i^2) = 2(1+2i-1) = 2(2i) = 4i$

$z^4 = z^2 z^2 = (4i)(4i) = 16i^2 = -16$

Hence $z^4 + 16 = -16 + 16 = 0$, as required.

**Notes**

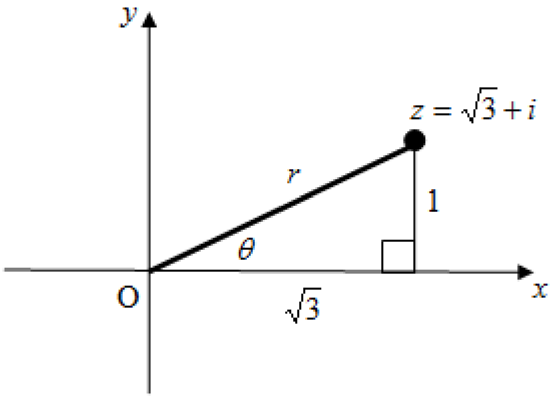
(1) An alternative way of expressing $z = \sqrt{2}(1+i)$ in polar form is to first express the complex number $1+i$ in polar form.

It can easily be shown that in polar form, $1+i = \sqrt{2}(\cos 45° + i\sin 45°)$,
hence $z = \sqrt{2}(1+i) = \sqrt{2} \times \sqrt{2}(\cos 45° + i\sin 45°) = 2(\cos 45° + i\sin 45°)$.

(2) The argument of a complex number can be expressed in degrees or radians.
In radians, $\theta = \frac{\pi}{4}$ leading to the polar form $z = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

(3) A binomial expansion could also be used to find $z^4$.

**8.**(a) $z = \sqrt{3} + i \quad \rightarrow \quad$ plot $\left(\sqrt{3}, 1\right)$




$$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\tan\theta = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

Hence $|z| = 2$ and $\arg(z) = 30^\circ$.

(b) $z = \sqrt{3} + i \quad \Rightarrow \quad \bar{z} = \sqrt{3} - i$

$$\frac{\bar{z}}{z} = \frac{\sqrt{3}-i}{\sqrt{3}+i} = \frac{\left(\sqrt{3}-i\right)\left(\sqrt{3}-i\right)}{\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)} = \frac{3-2\sqrt{3}i+i^2}{3-i^2} = \frac{3-2\sqrt{3}i-1}{3+1} = \frac{2-2\sqrt{3}i}{4} = \frac{1-\sqrt{3}i}{2} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

(c) polar form: $z = r(\cos\theta + i\sin\theta) \quad \Rightarrow \quad z = 2(\cos 30^\circ + i\sin 30^\circ)$

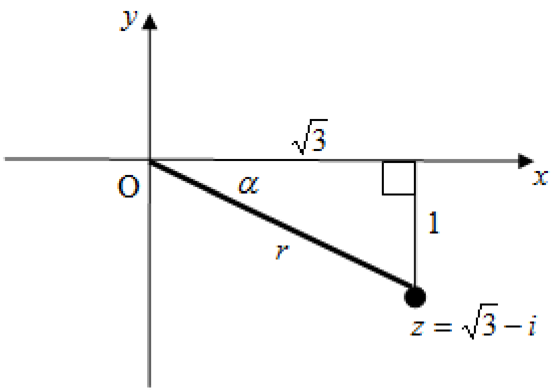
By de Moivre's theorem:
$$\begin{aligned} z^6 &= [2(\cos 30^\circ + i\sin 30^\circ)]^4 \\ &= 2^6\left(\cos 6(30^\circ) + i\sin 6(30^\circ)\right) \\ &= 64(\cos 180^\circ + i\sin 180^\circ) \\ &= 64(-1 + 0i) \\ &= 64(-1) \\ &= -64 \end{aligned}$$

**<u>Note</u>**

The argument of a complex number can be expressed in degrees or radians.

In radians, $\arg(z) = \dfrac{\pi}{6}$.

**9.**(a) $z = \sqrt{3} - i \quad \rightarrow \quad$ plot $\left(\sqrt{3}, -1\right)$




(b) To express the complex number $w = az$ in polar form, first express $z$ in polar form.

$$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\tan\alpha = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30° \quad \Rightarrow \quad \theta = -30°$$

polar form: $\quad z = r(\cos\theta + i\sin\theta) \quad \Rightarrow \quad z = 2\left(\cos(-30°) + i\sin(-30°)\right)$

Then $\quad w = az = a \times 2\left(\cos(-30°) + i\sin(-30°)\right)$
$$= 2a\left(\cos(-30°) + i\sin(-30°)\right)$$

(c) By de Moivre's theorem:
$$\begin{aligned} w^8 &= \left[2a\left(\cos(-30°) + i\sin(-30°)\right)\right]^8 \\ &= (2a)^8\left(\cos 8(-30°) + i\sin 8(-30°)\right) \\ &= 256a^8\left(\cos(-240°) + i\sin(-240°)\right) \\ &= 256a^8\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= 256a^8 \times \frac{1}{2}\left(-1 + \sqrt{3}i\right) \\ &= 128a^8\left(-1 + i\sqrt{3}\right) \end{aligned}$$

**<u>Notes</u>**

(1) Remember that $-180° < \arg(z) \le 180°$ or $-\pi < \arg(z) \le \pi$.

(2) The argument of a complex number can be expressed in degrees or radians.
In radians, $\theta = -\frac{\pi}{6}$ leading to the polar form $\quad w = 2a\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$.

**10.** de Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

By de Moivre's theorem: $$\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)^{11} = \cos 11\left(\frac{\pi}{18}\right) + i\sin 11\left(\frac{\pi}{18}\right)$$
$$= \cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}$$

By de Moivre's theorem: $$\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)^{4} = \cos 4\left(\frac{\pi}{36}\right) + i\sin 4\left(\frac{\pi}{36}\right)$$
$$= \cos\frac{\pi}{9} + i\sin\frac{\pi}{9}$$

Hence $$\frac{\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)^{4}} = \frac{\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}}{\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}}$$
$$= \frac{\left(\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}\right)\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)}{\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)}$$
$$= \frac{\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + i\sin\frac{11\pi}{18}\cos\frac{\pi}{9} - i\cos\frac{11\pi}{18}\sin\frac{\pi}{9} - i^2\sin\frac{11\pi}{18}\sin\frac{\pi}{9}}{\cos^2\frac{\pi}{9} - i^2\sin^2\frac{\pi}{9}}$$

$$= \frac{\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + i\sin\frac{11\pi}{18}\cos\frac{\pi}{9} - i\cos\frac{11\pi}{18}\sin\frac{\pi}{9} + \sin\frac{11\pi}{18}\sin\frac{\pi}{9}}{\cos^2\frac{\pi}{9} + \sin^2\frac{\pi}{9}}$$ [since $i^2 = -1$]

$$= \frac{\left(\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + \sin\frac{11\pi}{18}\sin\frac{\pi}{9}\right) + i\left(\sin\frac{11\pi}{18}\cos\frac{\pi}{9} - \cos\frac{11\pi}{18}\sin\frac{\pi}{9}\right)}{1}$$ [since $\sin^2\frac{\pi}{9} + \cos^2\frac{\pi}{9} = 1$]

$$= \left(\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + \sin\frac{11\pi}{18}\sin\frac{\pi}{9}\right) + i\left(\sin\frac{11\pi}{18}\cos\frac{\pi}{9} - \cos\frac{11\pi}{18}\sin\frac{\pi}{9}\right)$$

The real part of this complex number is $\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + \sin\frac{11\pi}{18}\sin\frac{\pi}{9}$.

From Higher, we have the addition formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

Hence $$\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + \sin\frac{11\pi}{18}\sin\frac{\pi}{9} = \cos\left(\frac{11\pi}{18} - \frac{\pi}{9}\right)$$

$$= \cos\left(\frac{11\pi}{18} - \frac{2\pi}{18}\right)$$

$$= \cos\frac{9\pi}{18}$$

$$= \cos\frac{\pi}{2}$$

$$= 0$$

This proves that the real part of $\dfrac{\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)^{4}}$ is zero, as required.

**11.** The quadratic equation $z^2 - \sqrt{8}z + 4 = 0$ must be solved using the quadratic formula.

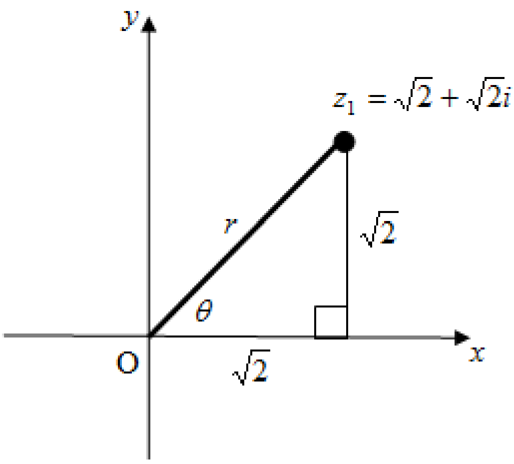
$a = 1$, $b = -\sqrt{8}$, $c = 4 \quad \Rightarrow \quad b^2 - 4ac = \left(-\sqrt{8}\right)^2 - 4(1)(4) = 8 - 16 = -8$

$b^2 - 4ac < 0$, so the equation has non-real roots.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\left(-\sqrt{8}\right) \pm \sqrt{-8}}{2(1)} = \frac{\sqrt{8} \pm \sqrt{8}i}{2} = \frac{2\sqrt{2} \pm 2\sqrt{2}i}{2} = \sqrt{2} \pm \sqrt{2}i$$

Denote the two roots by $z_1$ and $z_2$, where $z_1 = \sqrt{2} + \sqrt{2}i$ and $z_2 = \sqrt{2} - \sqrt{2}i$.

$z_1 = \sqrt{2} + \sqrt{2}i \quad \rightarrow \quad$ plot $\left(\sqrt{2}, \sqrt{2}\right)$




$$r = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = \sqrt{2+2} = \sqrt{4} = 2$$

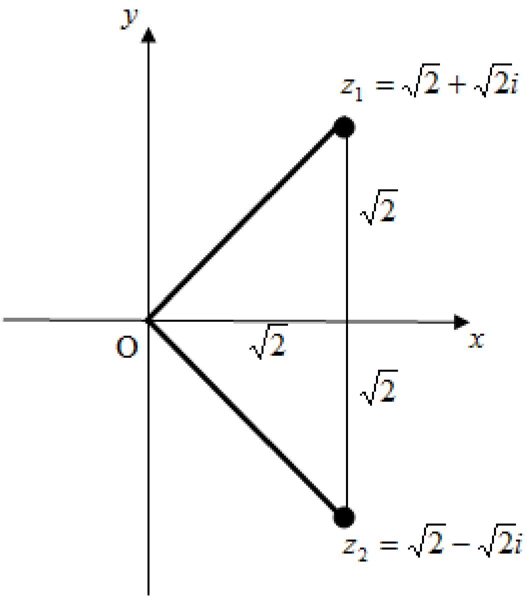
$$\tan\theta = \frac{\sqrt{2}}{\sqrt{2}} = 1 \quad \Rightarrow \quad \theta = \tan^{-1} 1 = 45°$$

Hence $|z_1| = 2$ and $\arg(z_1) = 45°$.

$z_2 = \sqrt{2} - \sqrt{2}i \quad \rightarrow \quad$ plot $\left(\sqrt{2}, -\sqrt{2}\right)$

Note that $z_2$ is the complex conjugate of $z_1$, hence $|z_2| = 2$ and $\arg(z_2) = -45°$.

The complex numbers $z_1$ and $z_2$ are plotted on the Argand diagram below.



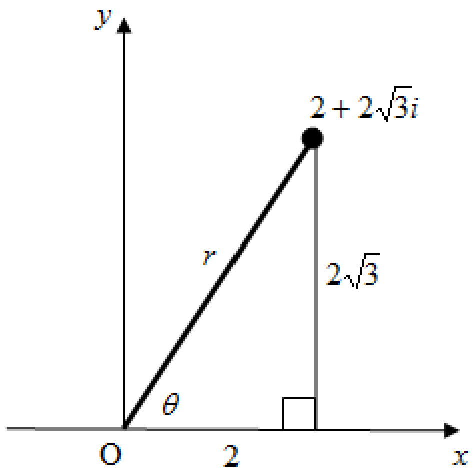

**<u>Notes</u>**

(1) Remember that $-180° < \arg(z) \leq 180°$ or $-\pi < \arg(z) \leq \pi$.

(2) The argument of a complex number can be expressed in degrees or radians.
In radians, $\arg(z_1) = \frac{\pi}{4}$ and $\arg(z_2) = -\frac{\pi}{4}$.

(3) If you know the modulus and argument of a complex number $z$, you can write down the modulus and argument of the complex conjugate $\bar{z}$ since $|\bar{z}| = |z|$ and $\arg(\bar{z}) = -\arg(z)$.

**12.**(a) $2+2\sqrt{3}i \quad \rightarrow \quad$ plot $\left(2, 2\sqrt{3}\right)$




$$r = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3} \quad \Rightarrow \quad \theta = \tan^{-1}\sqrt{3} = 60°$$

Hence the modulus of the complex number $2+2\sqrt{3}i$ is 4 and the argument is $60°$.

(b) We are required to find the two complex numbers $z$ such that $z^2 = 2+2\sqrt{3}i$.
We therefore need to find the two square roots of the complex number $2+2\sqrt{3}i$.

First express the complex number $2+2\sqrt{3}i$ in polar form.

polar form: $2+2\sqrt{3}i = r(\cos\theta + i\sin\theta) \quad \Rightarrow \quad 2+2\sqrt{3}i = 4(\cos 60° + i\sin 60°)$

Then $z^2 = 2+2\sqrt{3}i \quad \Rightarrow \quad z^2 = 4(\cos 60° + i\sin 60°)$

The first square root, $z_1$, is found using de Moivre's theorem:

$$z_1 = [4(\cos 60° + i\sin 60°)]^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(\cos\frac{1}{2}(60°) + i\sin\frac{1}{2}(60°)\right) = 2(\cos 30° + i\sin 30°)$$

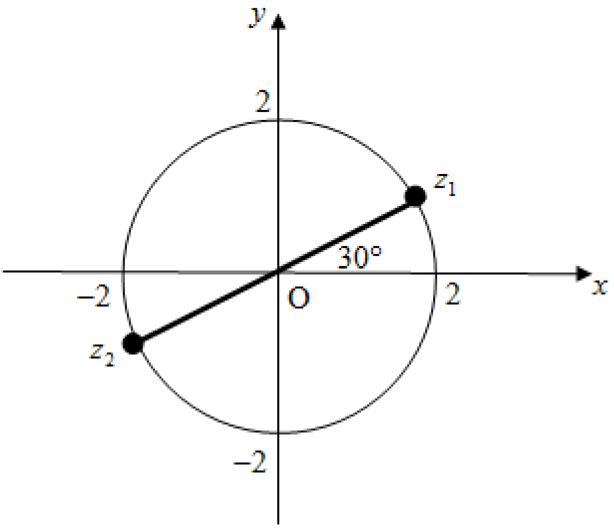
On an Argand diagram, the second square root, $z_2$, will lie directly opposite the first root, $z_1$, on a circle with centre O.

Hence $z_2 = 2\left(\cos(30° + 180°) + i\sin(30° + 180°)\right) = 2(\cos 210° + i\sin 210°)$.

The two square roots are $z_1 = 2(\cos 30° + i\sin 30°)$ and $z_2 = 2(\cos 210° + i\sin 210°)$.

(c) $z_1 = 2(\cos 30° + i \sin 30°)$ and $z_2 = 2(\cos 210° + i \sin 210°)$

The two square roots, $z_1$ and $z_2$, are plotted on the Argand diagram below.

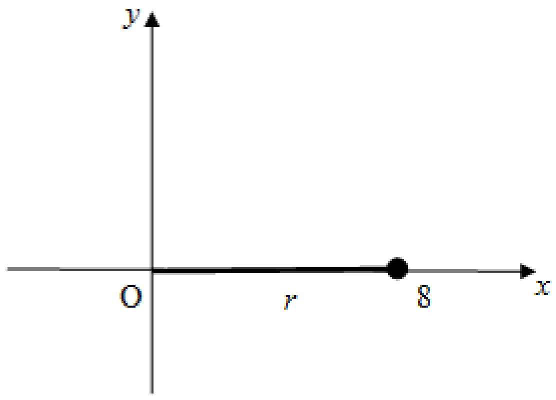



**Note**

The argument of a complex number can be expressed in degrees or radians.

In radians, the argument of the complex number $2 + 2\sqrt{3}i$ is $\frac{\pi}{3}$.

**13.**(a) $8 = 8 + 0i \quad \rightarrow \quad$ plot (8, 0)




Without calculation, clearly $r = 8$ and $\theta = 0$.

Hence $\quad 8 = r(\cos\theta + i\sin\theta) \quad$ where $\quad r = 8$ and $\theta = 0$.

(b) The three roots of the equation $z^3 = 8$ are the three cube roots of 8.

$z^3 = 8 \quad \Rightarrow \quad z^3 = 8(\cos 0 + i\sin 0)$

The first cube root, $z_1$, is found in polar form using de Moivre's theorem:

$$z_1 = [8(\cos 0 + i\sin 0)]^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(\cos\frac{1}{3}(0) + i\sin\frac{1}{3}(0)\right) = 2(\cos 0 + i\sin 0)$$

On an Argand diagram, the three cube roots will be equally spaced on a circle with centre O.

Working in degrees, $\dfrac{360}{3} = 120°$, hence $z_2 = 2\left(\cos(0 + 120°) + i\sin(0 + 120°)\right)$

$= 2(\cos 120° + i\sin 120°)$

and $\quad z_3 = 2\left(\cos(120° + 120°) + i\sin(120° + 120°)\right)$

$= 2(\cos 240° + i\sin 240°)$

The three cube roots can now be expressed in Cartesian form:

$$z_1 = 2(\cos 0 + i\sin 0) = 2(1+0i) = 2(1) = 2$$

$$z_2 = 2(\cos 120° + i\sin 120°) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$$

$$z_3 = 2(\cos 240° + i\sin 240°) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - \sqrt{3}i$$

(c) (i)

$$\begin{aligned} z_1 + z_2 + z_3 &= 2 + \left(-1+\sqrt{3}i\right) + \left(-1-\sqrt{3}i\right) \\ &= 2 - 1 + \sqrt{3}i - 1 - \sqrt{3}i \\ &= 0 \end{aligned}$$

(ii) **<u>Method 1</u>**

Note that $z_1$, $z_2$ and $z_3$ are solutions of the equation $z^3 = 8$, hence $z_1{}^3 = 8$, $z_2{}^3 = 8$ and $z_3{}^3 = 8$.

Then

$$\begin{aligned} z_1{}^6 + z_2{}^6 + z_3{}^6 &= \left(z_1{}^3\right)^2 + \left(z_2{}^3\right)^2 + \left(z_3{}^3\right)^2 \\ &= 8^2 + 8^2 + 8^2 \\ &= 64 + 64 + 64 \\ &= 192 \end{aligned}$$

**<u>Method 2</u>**

$$z_1 = 2 \quad \Rightarrow \quad z_1{}^6 = 2^6 = 64$$

$z_2{}^6$ and $z_3{}^6$ can be found using de Moivre's theorem.

$$\begin{aligned} z_2 = 2(\cos 120° + i\sin 120°) \quad \Rightarrow \quad z_2{}^6 &= \left[2(\cos 120° + i\sin 120°)\right]^6 \\ &= 2^6\left(\cos 6(120°) + i\sin 6(120°)\right) \\ &= 64(\cos 720° + i\sin 720°) \\ &= 64(1+0i) \\ &= 64(1) \\ &= 64 \end{aligned}$$

$z_3 = 2(\cos 240° + i\sin 240°) \quad \Rightarrow \quad z_3^{\ 6} = [2(\cos 240° + i\sin 240°)]^6$

$$\begin{aligned} &= 2^6\left(\cos 6(240°) + i\sin 6(240°)\right) \\ &= 64(\cos 1440° + i\sin 1440°) \\ &= 64(1+0i) \\ &= 64(1) \\ &= 64 \end{aligned}$$

Hence $z_1^{\ 6} + z_2^{\ 6} + z_3^{\ 6} = 64 + 64 + 64 = 192$.

**<u>Note</u>**

Working in radians gives, in polar form,

$$z_2 = 2\left(\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}\right) \quad \text{and} \quad z_2 = 2\left(\cos\frac{4\pi}{3} + \sin\frac{4\pi}{3}\right).$$

**14.** Let $z = x + yi$.

Then $|z+i| = 2 \quad \Rightarrow \quad |(x+yi)+i| = 2$

$$\begin{aligned} &\Rightarrow \quad |x+(y+1)i| = 2 \\ &\Rightarrow \quad \sqrt{x^2+(y+1)^2} = 2 \\ &\Rightarrow \quad x^2+(y+1)^2 = 4 \end{aligned}$$

The equation of the locus of $z$ in the complex plane is $x^2+(y+1)^2 = 4$.
The locus of $z$ is the set of points on the circle with centre (0, −1) and radius 2.

**<u>Notes</u>**

(1) The modulus of the complex number $a+bi$ is given by $|a+bi| = \sqrt{a^2+b^2}$.

(2) The equation of the circle with centre ($a$, $b$) and radius $r$ is $(x-a)^2+(y-b)^2 = r^2$.

**15.**(a) Let $z = x + yi$.

Then $|z-1| = 3 \Rightarrow |(x+yi)-1| = 3$

$\Rightarrow |(x-1)+yi| = 3$

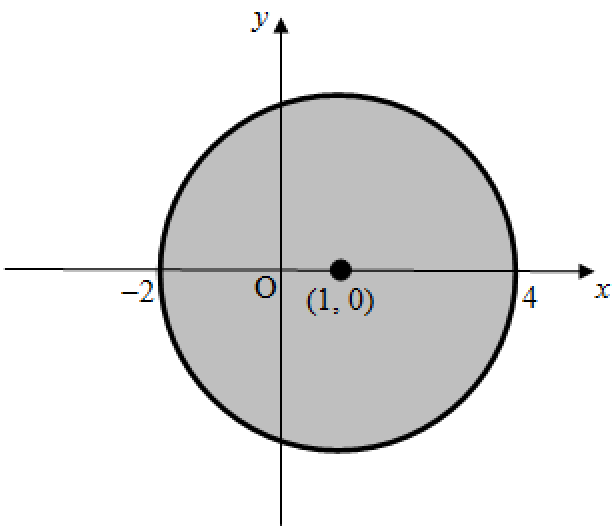
$\Rightarrow \sqrt{(x-1)^2 + y^2} = 3$

$\Rightarrow (x-1)^2 + y^2 = 9$

The equation of the locus of $z$ in the complex plane is $(x-1)^2 + y^2 = 9$.
The locus of $z$ is the set of points on the circle with centre (1, 0) and radius 3.

(b) If $|z-1| \le 3$, the locus of $z$ is the set of points on or inside the circle with centre (1, 0) and radius 3.

The region in the complex plane given by $|z-1| \le 3$ is shaded in the diagram below.




**Notes**

(1) The modulus of the complex number $a + bi$ is given by $|a+bi| = \sqrt{a^2 + b^2}$.

(2) The equation of the circle with centre ($a$, $b$) and radius $r$ is $(x-a)^2 + (y-b)^2 = r^2$.

**16.** Let $z = x + yi$ .

Then $|z-2| = |z+i| \Rightarrow |(x+yi)-2| = |(x+yi)+i|$

$$\Rightarrow |(x-2)+yi| = |x+(y+1)i|$$

$$\Rightarrow \sqrt{(x-2)^2+y^2} = \sqrt{x^2+(y+1)^2}$$

$$\Rightarrow (x-2)^2+y^2 = x^2+(y+1)^2$$

$$\Rightarrow x^2-4x+4+y^2 = x^2+y^2+2y+1$$

$$\Rightarrow -4x+4 = 2y+1$$

$$\Rightarrow -4x-2y+3 = 0$$

$$\Rightarrow 4x+2y-3 = 0$$

Hence $ax+by+c=0$ where $a=4$ , $b=2$ and $c=-3$.

The equation of the locus of *z* is of the form $ax+by+c=0$, hence the locus of *z* is a straight line.

The locus of *z* is the set of points on the straight line with equation $4x+2y-3=0$ .

To sketch the straight line with equation $4x+2y-3=0$, find the points where the line crosses the *x*-axis and *y*-axis.

The line crosses the *y*-axis when $x=0 \Rightarrow 4(0)+2y-3=0$

$$\Rightarrow 2y-3=0$$

$$\Rightarrow 2y=3$$

$$\Rightarrow y=\frac{3}{2} \Rightarrow \text{point } \left(0,\frac{3}{2}\right)$$

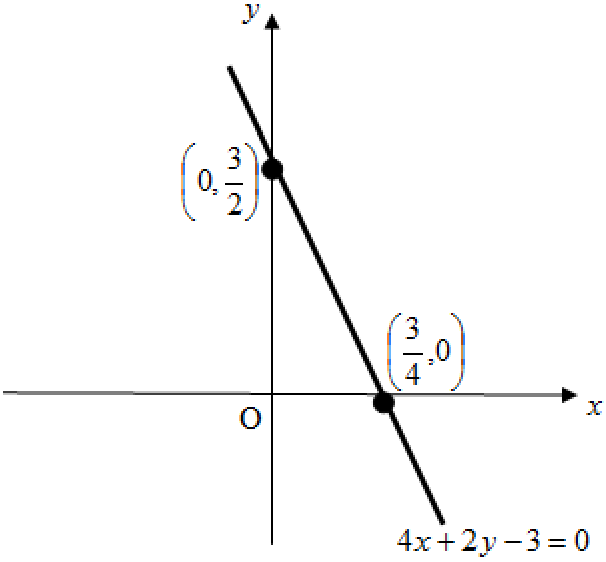
The line crosses the *x*-axis when $y=0 \Rightarrow 4x+2(0)-3=0$

$$\Rightarrow 4x-3=0$$

$$\Rightarrow 4x=3$$

$$\Rightarrow x=\frac{3}{4} \Rightarrow \text{point } \left(\frac{3}{4},0\right)$$

The locus of $z$ is shown on the Argand diagram below.




**Note**

The modulus of the complex number $a + bi$ is given by $|a + bi| = \sqrt{a^2 + b^2}$ .

**17.** Let $z = x + yi$ .

Then $|z| = |z - 2 + 2i| \Rightarrow |x + yi| = |(x + yi) - 2 + 2i|$

$$\Rightarrow |x + yi| = |(x - 2) + (y + 2)i|$$

$$\Rightarrow \sqrt{x^2 + y^2} = \sqrt{(x-2)^2 + (y+2)^2}$$

$$\Rightarrow x^2 + y^2 = (x-2)^2 + (y+2)^2$$

$$\Rightarrow x^2 + y^2 = x^2 - 4x + 4 + y^2 + 4y + 4$$

$$\Rightarrow 0 = -4x + 4 + 4y + 4$$

$$\Rightarrow 0 = -4x + 4y + 8$$

$$\Rightarrow 4x - 4y - 8 = 0$$

$$\Rightarrow x - y - 2 = 0$$

$$\Rightarrow y = x - 2$$

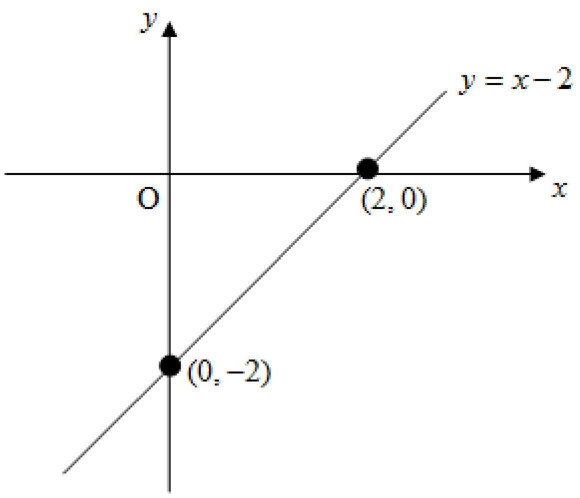
The locus of *z* is the set of points on the straight line with equation $y = x - 2$ .

To sketch the straight line with equation $y = x - 2$ , find the points where the line crosses the *x*-axis and *y*-axis.

The line crosses the *y*-axis when $x = 0 \Rightarrow y = 0 - 2 = -2 \Rightarrow$ point (0, −2)

The line crosses the *x*-axis when $y = 0 \Rightarrow 0 = x - 2 \Rightarrow x = 2 \Rightarrow$ point (2, 0)

The locus of *z* is shown on the Argand diagram below.




**<u>Note</u>**

The modulus of the complex number $a + bi$ is given by $|a + bi| = \sqrt{a^2 + b^2}$ .

**18.**(a) Let $z = x + yi$ .

Then $|z+i| = 1 \quad \Rightarrow \quad |(x+yi)+i| = 1$

$$\Rightarrow \quad |x + (y+1)i| = 1$$

$$\Rightarrow \quad \sqrt{x^2 + (y+1)^2} = 1$$

$$\Rightarrow \quad x^2 + (y+1)^2 = 1$$

The equation of the locus of $z$ in the complex plane is $x^2 + (y+1)^2 = 1$.

The locus of $z$ is the set of points on the circle with centre (0, −1) and radius 1.

(b) Let $z = x + yi$ .

Then $|z-1| = |z+5| \quad \Rightarrow \quad |(x+yi)-1| = |(x+yi)+5|$

$$\Rightarrow \quad |(x-1)+yi| = |(x+5)+yi|$$

$$\Rightarrow \quad \sqrt{(x-1)^2 + y^2} = \sqrt{(x+5)^2 + y^2}$$

$$\Rightarrow \quad (x-1)^2 + y^2 = (x+5)^2 + y^2$$

$$\Rightarrow \quad x^2 - 2x + 1 + y^2 = x^2 + 10x + 25 + y^2$$

$$\Rightarrow \quad -2x + 1 = 10x + 25$$

$$\Rightarrow \quad -12x - 24 = 0$$

$$\Rightarrow \quad 12x + 24 = 0$$

$$\Rightarrow \quad x + 2 = 0$$

$$\Rightarrow \quad x = -2$$

The equation of the locus of $z$ in the complex plane is $x = -2$.

The locus of $z$ is the set of points on the vertical line with equation $x = -2$.

**Notes**

(1) The modulus of the complex number $a + bi$ is given by $|a+bi| = \sqrt{a^2 + b^2}$ .

(2) The equation of the circle with centre ($a$, $b$) and radius $r$ is $(x-a)^2 + (y-b)^2 = r^2$.

**19.**(a) Let $z = x + yi$.

Then $z^2 = |z|^2 - 4 \Rightarrow (x+yi)^2 = |x+yi|^2 - 4$

$$\Rightarrow x^2 + 2xyi + y^2 i^2 = \left(\sqrt{x^2+y^2}\right)^2 - 4$$
$$\Rightarrow x^2 + 2xyi - y^2 = x^2 + y^2 - 4$$
$$\Rightarrow (x^2 - y^2) + 2xyi = x^2 + y^2 - 4$$

Equating real parts $\Rightarrow x^2 - y^2 = x^2 + y^2 - 4$

$$\Rightarrow -y^2 = y^2 - 4$$
$$\Rightarrow 2y^2 = 4$$
$$\Rightarrow y^2 = 2$$
$$\Rightarrow y = \pm\sqrt{2}$$

Equating imaginary parts $\Rightarrow 2xy = 0 \Rightarrow xy = 0$ ...(1)

Substitute $y = \sqrt{2}$ into equation (1) $\Rightarrow x\left(\sqrt{2}\right) = 0 \Rightarrow x = 0 \Rightarrow z = 0 + \sqrt{2}i = \sqrt{2}i$

Substitute $y = -\sqrt{2}$ into equation (1) $\Rightarrow x\left(-\sqrt{2}\right) = 0 \Rightarrow x = 0$

$$\Rightarrow z = 0 + \left(-\sqrt{2}i\right) = -\sqrt{2}i$$

Hence the two solutions of the equation are $z = \sqrt{2}i$ and $z = -\sqrt{2}i$.

(b) Let $z = x + yi$.

Then $z^2 = i(|z|^2 - 4) \Rightarrow (x+yi)^2 = i\left(|x+yi|^2 - 4\right)$

$$\Rightarrow (x^2 - y^2) + 2xyi = i(x^2 + y^2 - 4)$$ [using the working in (a)]

Equating real parts $\Rightarrow x^2 - y^2 = 0 \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$

Equating imaginary parts $\Rightarrow 2xy = x^2 + y^2 - 4$ ...(2)

Substitute $y = x$ into equation (2) $\Rightarrow 2x(x) = x^2 + x^2 - 4$

$$\Rightarrow 2x^2 = 2x^2 - 4$$
$$\Rightarrow 0 = -4$$

This equation has no solution, hence $y \neq x$.

Substitute $y = -x$ into equation (2) $\Rightarrow$ $2x(-x) = x^2 + (-x)^2 - 4$

$$\Rightarrow \quad -2x^2 = x^2 + x^2 - 4$$

$$\Rightarrow \quad -2x^2 = 2x^2 - 4$$

$$\Rightarrow \quad 4x^2 = 4$$

$$\Rightarrow \quad x^2 = 1$$

$$\Rightarrow \quad x = \pm 1$$

$y = -x$, so when $x = 1$, $y = -1$ $\Rightarrow$ $z = 1 - i$

$y = -x$, so when $x = -1$, $y = 1$ $\Rightarrow$ $z = -1 + i$

Hence the two solutions of the equation are $z = 1 - i$ and $z = -1 + i$ .

**20.**(a) $z = \cos\theta + i\sin\theta$

By the binomial theorem:

$$z^4 = (\cos\theta + i\sin\theta)^4$$
$$= \binom{4}{0}(\cos\theta)^4(i\sin\theta)^0 + \binom{4}{1}(\cos\theta)^3(i\sin\theta)^1 + \binom{4}{2}(\cos\theta)^2(i\sin\theta)^2 + \binom{4}{3}(\cos\theta)^1(i\sin\theta)^3$$
$$+ \binom{4}{4}(\cos\theta)^0(i\sin\theta)^4$$
$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i^2\sin^2\theta) + 4\cos\theta(i^3\sin^3\theta) + i^4\sin^4\theta$$
$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(-1)\sin^2\theta + 4\cos\theta(-i)\sin^3\theta + 1\sin^4\theta$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$
$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

real part of $z^4 = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

imaginary part of $z^4 = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

(b) By de Moivre's theorem: $z^4 = (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

(c) Equating real parts of $z^4$ $\Rightarrow$ $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(d) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

To express $\cos 4\theta$ in the form $k(\cos^m\theta - \cos^n\theta) + p$, we must write $\sin^2\theta$ and $\sin^4\theta$ in terms of $\cos\theta$.

$\sin^2\theta + \cos^2\theta = 1 \quad \Rightarrow \quad \sin^2\theta = 1 - \cos^2\theta$

$\sin^4\theta = \sin^2\theta\sin^2\theta = (1-\cos^2\theta)(1-\cos^2\theta) = 1 - 2\cos^2\theta + \cos^4\theta$

Hence
$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$
$$= 8(\cos^4\theta - \cos^2\theta) + 1$$

Hence $\cos 4\theta = k(\cos^m\theta - \cos^n\theta) + p$ where $k = 8$, $m = 4$, $n = 2$ and $p = 1$.

**21.**(a) Let $z = \cos\theta + i\sin\theta$.

By de Moivre's theorem: $z^5 = (\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

By the binomial theorem:

$$\begin{aligned} z^5 &= (\cos\theta + i\sin\theta)^5 \\ &= \binom{5}{0}(\cos\theta)^5(i\sin\theta)^0 + \binom{5}{1}(\cos\theta)^4(i\sin\theta)^1 + \binom{5}{2}(\cos\theta)^3(i\sin\theta)^2 + \binom{5}{3}(\cos\theta)^2(i\sin\theta)^3 \\ &\quad + \binom{5}{4}(\cos\theta)^1(i\sin\theta)^4 + \binom{5}{5}(\cos\theta)^0(i\sin\theta)^5 \end{aligned}$$

$$= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i^2\sin^2\theta) + 10\cos^2\theta(i^3\sin^3\theta) + 5\cos\theta(i^4\sin^4\theta) + i^5\sin^5\theta$$

$$= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(-1)\sin^2\theta + 10\cos^2\theta(-i)\sin^3\theta + 5\cos\theta(1)\sin^4\theta + i\sin^5\theta$$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Equating imaginary parts of $z^5$ $\Rightarrow$ $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Hence $\sin 5\theta = k\cos^4\theta\sin\theta + l\cos^2\theta\sin^3\theta + m\sin^5\theta$ where $k = 5$, $l = -10$ and $m = 1$.

(b) $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

To express $\sin 5\theta$ entirely in terms of $\sin\theta$, we must write $\cos^4\theta$ and $\sin^2\theta$ in terms of $\sin\theta$.

$\sin^2\theta + \cos^2\theta = 1$ $\Rightarrow$ $\cos^2\theta = 1 - \sin^2\theta$

$\cos^4\theta = \cos^2\theta\cos^2\theta = (1 - \sin^2\theta)(1 - \sin^2\theta) = 1 - 2\sin^2\theta + \sin^4\theta$

Hence
$$\begin{aligned} \sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta \\ &= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta(1 - \sin^2\theta) + \sin^5\theta \\ &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta \end{aligned}$$

$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$ *or* $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

**22.**(a) Let $z = \cos\theta + i\sin\theta$.

By the binomial theorem:

$$
\begin{aligned}
z^4 &= (\cos\theta + i\sin\theta)^4 \\
&= \binom{4}{0}(\cos\theta)^4(i\sin\theta)^0 + \binom{4}{1}(\cos\theta)^3(i\sin\theta)^1 + \binom{4}{2}(\cos\theta)^2(i\sin\theta)^2 + \binom{4}{3}(\cos\theta)^1(i\sin\theta)^3 \\
&\quad + \binom{4}{4}(\cos\theta)^0(i\sin\theta)^4 \\
&= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i^2\sin^2\theta) + 4\cos\theta(i^3\sin^3\theta) + i^4\sin^4\theta \\
&= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(-1)\sin^2\theta + 4\cos\theta(-i)\sin^3\theta + 1\sin^4\theta \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)
\end{aligned}
$$

Hence $z^4 = u + iv$ where $u = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
and $v = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

(b) By de Moivre's theorem: $z^4 = (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

(c) First find an expression for $\cos 4\theta$.

Equating real parts of $z^4 \Rightarrow \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

To show that $\dfrac{\cos 4\theta}{\cos^2\theta} = p\cos^2\theta + q\sec^2\theta + r = p\cos^2\theta + \dfrac{q}{\cos^2\theta} + r$, note that this expression involves $\cos\theta$ only which suggests that we write $\cos 4\theta$ entirely in terms of $\cos\theta$.

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

To express $\cos 4\theta$ entirely in terms of $\cos\theta$, we must write $\sin^2\theta$ and $\sin^4\theta$ in terms of $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1 \quad \Rightarrow \quad \sin^2\theta = 1 - \cos^2\theta$$

$$\sin^4\theta = \sin^2\theta\sin^2\theta = (1-\cos^2\theta)(1-\cos^2\theta) = 1 - 2\cos^2\theta + \cos^4\theta$$

Hence
$$
\begin{aligned}
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
&= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + 1 - 2\cos^2\theta + \cos^4\theta \\
&= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\
&= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}
$$

$$\frac{\cos 4\theta}{\cos^2\theta} = \frac{8\cos^4\theta - 8\cos^2\theta + 1}{\cos^2\theta}$$

$$= \frac{8\cos^4\theta}{\cos^2\theta} - \frac{8\cos^2\theta}{\cos^2\theta} + \frac{1}{\cos^2\theta}$$

$$= 8\cos^2\theta - 8 + \sec^2\theta \qquad \text{[since } \sec^2\theta = \frac{1}{\cos^2\theta}\text{ ]}$$

$$= 8\cos^2\theta + \sec^2\theta - 8$$

Hence $\frac{\cos 4\theta}{\cos^2\theta} = p\cos^2\theta + q\sec^2\theta + r$ where $p = 8$, $q = 1$ and $r = -8$.

**23.**(a) $z = \cos\theta + i\sin\theta$

By de Moivre's theorem: $z^4 = (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

By the binomial theorem:

$$
\begin{aligned}
z^4 &= (\cos\theta + i\sin\theta)^4 \\
&= \binom{4}{0}(\cos\theta)^4(i\sin\theta)^0 + \binom{4}{1}(\cos\theta)^3(i\sin\theta)^1 + \binom{4}{2}(\cos\theta)^2(i\sin\theta)^2 + \binom{4}{3}(\cos\theta)^1(i\sin\theta)^3 \\
&\quad + \binom{4}{4}(\cos\theta)^0(i\sin\theta)^4 \\
&= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i^2\sin^2\theta) + 4\cos\theta(i^3\sin^3\theta) + i^4\sin^4\theta \\
&= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(-1)\sin^2\theta + 4\cos\theta(-i)\sin^3\theta + 1\sin^4\theta \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)
\end{aligned}
$$

(i) Equating real parts of $z^4$ $\Rightarrow$ $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii) Equating imaginary parts of $z^4$ $\Rightarrow$ $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

(b) $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ [now divide all terms by $\cos^4\theta$]

$$
= \frac{\dfrac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \dfrac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\dfrac{\cos^4\theta}{\cos^4\theta} - \dfrac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}
$$

$$
= \frac{\dfrac{4\sin\theta}{\cos\theta} - \dfrac{4\sin^3\theta}{\cos^3\theta}}{1 - \dfrac{6\sin^2\theta}{\cos^2\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}
$$

$$
= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}
$$

**24.**(a) $w = \cos\theta + i\sin\theta$

$$\frac{1}{w} = \frac{1}{\cos\theta + i\sin\theta} = \frac{1(\cos\theta - i\sin\theta)}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{1} \qquad [\text{since } \sin^2\theta + \cos^2\theta = 1]$$

$$= \cos\theta - i\sin\theta$$

(b) By de Moivre's theorem: $w^k = (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

$$w^{-k} = \frac{1}{w^k} = \frac{1}{\cos k\theta + i\sin k\theta} = \frac{1(\cos k\theta - i\sin k\theta)}{(\cos k\theta + i\sin k\theta)(\cos k\theta - i\sin k\theta)}$$

$$= \frac{\cos k\theta - i\sin k\theta}{\cos^2 k\theta - i^2\sin^2 k\theta}$$

$$= \frac{\cos k\theta - i\sin k\theta}{\cos^2 k\theta + \sin^2 k\theta}$$

$$= \frac{\cos k\theta - i\sin k\theta}{1} \qquad [\text{since } \sin^2 k\theta + \cos^2 k\theta = 1]$$

$$= \cos k\theta - i\sin k\theta$$

Hence $w^k + w^{-k} = (\cos k\theta + i\sin k\theta) + (\cos k\theta + i\sin k\theta)$

$$= \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta$$

$$= 2\cos k\theta$$

(c) By the binomial theorem:

$$(w + w^{-1})^4 = \binom{4}{0} w^4 (w^{-1})^0 + \binom{4}{1} w^3 (w^{-1})^1 + \binom{4}{2} w^2 (w^{-1})^2 + \binom{4}{3} w^1 (w^{-1})^3 + \binom{4}{4} w^0 (w^{-1})^4$$

$$= w^4 + 4w^3 w^{-1} + 6w^2 w^{-2} + 4w^1 w^{-3} + w^{-4}$$

$$= w^4 + 4w^2 + 6w^0 + 4w^{-2} + w^{-4}$$

$$= w^4 + 4w^2 + 6 + 4w^{-2} + w^{-4}$$

$$(w + w^{-1})^4 = w^4 + 4w^2 + 6 + 4w^{-2} + w^{-4}$$

The LHS and RHS of the above equation can be simplified using the fact that $w^k + w^{-k} = 2\cos k\theta$ from (b).

LHS $= (w + w^{-1})^4 = (2\cos\theta)^4 = 16\cos^4\theta$ [since $w^1 + w^{-1} = 2\cos 1\theta = 2\cos\theta$]

RHS $= w^4 + 4w^2 + 6 + 4w^{-2} + w^{-4}$

$= (w^4 + w^{-4}) + 4(w^2 + w^{-2}) + 6$ [grouping terms of the form $w^k + w^{-k}$]

$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$ [using $w^k + w^{-k} = 2\cos k\theta$]

$= 2\cos 4\theta + 8\cos 2\theta + 6$

Hence $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6 \quad \Rightarrow \quad \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$