ADVANCED HIGHER MATHEMATICS

Exam Questions on Complex Numbers 2

1. Given that $z = 2 + 3i$, plot on an Argand diagram the points that represent the complex numbers *z*, $\bar{z}$ and z^2, where $\bar{z}$ is the complex conjugate of *z*.

2. Express $z = \frac{(1+2i)^2}{7-i}$ in the form $a + bi$ where *a* and *b* are real numbers.

Show *z* on an Argand diagram and evaluate $|z|$ and arg (*z*).

3. Express the complex number $\frac{1+3i}{1-2i}$ in the form $x + iy$, where *x* and *y* are real numbers.

Determine the modulus and argument of this complex number.

4. Determine the modulus and argument of the complex number $\frac{1}{1-i}$.

5. Given that $z = 1 - \sqrt{3}i$, write down $\bar{z}$ and express $(\bar{z})^2$ in polar form.

6. Express the complex number $z = -i + \frac{1}{1-i}$ in the form $z = x + yi$, stating the values of *x* and *y*.

Find the modulus and argument of *z* and plot *z* and $\bar{z}$ on an Argand diagram, where $\bar{z}$ is the complex conjugate of *z*.

7. Write the complex number $z = \sqrt{2}(1+i)$ in polar form and verify that *z* satisfies the equation $z^4 + 16 = 0$.

8. (a) Plot the complex number $z = \sqrt{3} + i$ on an Argand diagram and find the modulus and argument of *z*.

(b) Find $\frac{\bar{z}}{z}$ where $\bar{z}$ denotes the complex conjugate of *z*.

(c) Use de Moivre's theorem to find z^6.

9. Let $z = \sqrt{3} - i$.

(a) Plot z on an Argand diagram.

(b) Let $w = az$, where $a > 0$, $a \in \mathbf{R}$.
Express w in polar form.

(c) Express w^8 in the form $ka^n(x + i\sqrt{y})$, where $k, x, y \in \mathbf{Z}$.

10. De Moivre's Theorem states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for any integer n.

Use de Moivre's theorem to show that the real part of $\dfrac{\left(\cos\dfrac{\pi}{18} + i\sin\dfrac{\pi}{18}\right)^{11}}{\left(\cos\dfrac{\pi}{36} + i\sin\dfrac{\pi}{36}\right)^{4}}$ is zero.

11. Solve the equation

$$z^2 - \sqrt{8}z + 4 = 0$$

for the complex number z.

Give the modulus and argument of each of the roots and illustrate them on an Argand diagram.

12. (a) Find the modulus and argument of the complex number $2 + 2\sqrt{3}i$ and plot it on an Argand diagram.

(b) Find the two complex numbers z such that

$$z^2 = 2 + 2\sqrt{3}i,$$

expressing each solution in the form $z = r(\cos\theta + i\sin\theta)$.

(c) Plot the complex numbers z on an Argand diagram.

13. (a) Find the values of r and θ such that $8 = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$.

(b) Hence obtain the three roots of the equation $z^3 = 8$ in Cartesian form.

(c) Denoting the roots of $z^3 = 8$ by z_1, z_2 and z_3:

(i) state the value of $z_1 + z_2 + z_3$;

(ii) obtain the value of $z_1^6 + z_2^6 + z_3^6$.

14. Identify the locus in the complex plane given by $|z + i| = 2$.

15. (a) Identify the locus in the complex plane given by $|z - 1| = 3$.

(b) Show in a diagram the region given by $|z - 1| \le 3$.

16. Given that $|z - 2| = |z + i|$, where $z = x + yi$, show that $ax + by + c = 0$ for suitable values of a, b and c.

Indicate on an Argand diagram the locus of complex numbers z which satisfy $|z - 2| = |z + i|$.

17. Given $z = x + yi$, sketch the locus in the complex plane given by $|z| = |z - 2 + 2i|$.

18. Determine the loci in the complex plane given by:

(a) $|z + i| = 1$

(b) $|z - 1| = |z + 5|$

19. By writing z in the form $x + yi$:

(a) solve the equation $z^2 = |z|^2 - 4$;

(b) find the solutions to the equation $z^2 = i(|z|^2 - 4)$.

20. Let $z = \cos\theta + i\sin\theta$.

(a) Use a binomial expansion to show that the real part of z^4 is

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ.

(b) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ.

(c) Use your answers to (a) and (b) to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(d) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$ where *k*, *m*, *n* and *p* are integers. State the values of *k*, *m*, *n* and *p*.

21. De Moivre's Theorem states that

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

for any integer *n*.

(a) By using De Moivre's Theorem and a binomial expansion, show that $\sin 5\theta$ can be expressed in the form

$$k\cos^4\theta\sin\theta + l\cos^2\theta\sin^3\theta + m\sin^5\theta$$

for some real values of *k*, *l* and *m*. Write down the values of *k*, *l* and *m*.

(b) Hence find an expression for $\sin 5\theta$ entirely in terms of $\sin\theta$.

22. Let $z = \cos\theta + i\sin\theta$.

(a) Use a binomial expansion to express z^4 in the form $u + iv$, where u and v are expressions involving $\sin\theta$ and $\cos\theta$.

(b) Use de Moivre's theorem to write down a second expression for z^4.

(c) Using the results of (a) and (b), show that

$$\frac{\cos 4\theta}{\cos^2\theta} = p\cos^2\theta + q\sec^2\theta + r\text{, where } -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

stating the values of *p*, *q* and *r*.

23. (a) Given $z = \cos\theta + i\sin\theta$, use de Moivre's theorem and the binomial theorem to show that:

(i) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

(b) Hence show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

24. (a) Given that $w = \cos\theta + i\sin\theta$, show that $\dfrac{1}{w} = \cos\theta - i\sin\theta$.

(b) Use de Moivre's theorem to prove that $w^k + w^{-k} = 2\cos k\theta$, where k is a natural number.

(c) Expand $(w + w^{-1})^4$ by the binomial theorem and hence show that

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$