

Solutions to Exam Questions on Complex Numbers 1

$$1.(a) \quad z_2 = p - 6i \Rightarrow \bar{z}_2 = p + 6i$$

$$\begin{aligned} z_1 \bar{z}_2 &= (2 + 3i)(p + 6i) = 2p + 12i + 3pi + 18i^2 \\ &= 2p + 12i + 3pi - 18 \quad [\text{since } i^2 = -1] \\ &= (2p - 18) + (12 + 3p)i \end{aligned}$$

$$(b) \quad z_1 \bar{z}_2 \text{ is a real number when } 12 + 3p = 0 \Rightarrow 3p = -12 \Rightarrow p = -4$$

$$2. \quad \text{Let } z = a + bi \Rightarrow \bar{z} = a - bi$$

$$\begin{aligned} z + 2i\bar{z} &= 8 + 7i \Rightarrow a + bi + 2i(a - bi) = 8 + 7i \\ &\Rightarrow a + bi + 2ai - 2bi^2 = 8 + 7i \\ &\Rightarrow a + bi + 2ai + 2b = 8 + 7i \quad [\text{since } i^2 = -1] \\ &\Rightarrow (a + 2b) + (b + 2a)i = 8 + 7i \end{aligned}$$

$$\text{Equating real parts } \Rightarrow a + 2b = 8 \quad \dots(1)$$

$$\text{Equating imaginary parts } \Rightarrow b + 2a = 7 \Rightarrow 2a + b = 7 \quad \dots(2)$$

Solving equations (1) and (2) gives $a = 2$ and $b = 3$.

Hence $z = 2 + 3i$.

3. $z = -1 + 2i$ is given to be a root, hence the complex conjugate $z = -1 - 2i$ is also a root.

First find the factor corresponding to each root.

Remember that the factor corresponding to the root $z = \alpha$ is $(z - \alpha)$.

$$\text{root } z = -1 + 2i \Rightarrow \text{factor } (z - (-1 + 2i))$$

$$\text{root } z = -1 - 2i \Rightarrow \text{factor } (z - (-1 - 2i))$$

These two factors can be multiplied together to give a quadratic factor as follows.

$$\begin{aligned}(z - (-1 + 2i))(z - (-1 - 2i)) &= ((z + 1) - 2i)((z + 1) + 2i) \\ &= (z + 1)^2 - 4i^2 \\ &= z^2 + 2z + 1 + 4 \quad [\text{since } i^2 = -1] \\ &= z^2 + 2z + 5\end{aligned}$$

The other factor can be found using algebraic long division.

$$\begin{array}{r} z + 3 \\ \hline z^2 + 2z + 5 \overline{) z^3 + 5z^2 + 11z + 15} \\ \underline{z^3 + 2z^2 + 5z} \\ 3z^2 + 6z + 15 \\ \underline{3z^2 + 6z + 15} \\ 0 \end{array}$$

$$\text{Hence } (z^2 + 2z + 5)(z + 3) = 0$$

$$\text{and the remaining root comes from the equation } z + 3 = 0 \Rightarrow z = -3$$

The roots are $z = -1 + 2i$, $z = -1 - 2i$ and $z = -3$.

8. Let $z = i$.

$$z^2 = i^2 = -1 \qquad z^3 = zz^2 = i(-1) = -i \qquad z^4 = z^2z^2 = (-1)(-1) = 1$$

$$\begin{aligned} \text{Hence } z^4 + 4z^3 + 3z^2 + 4z + 2 &= 1 + 4(-i) + 3(-1) + 4i + 2 \\ &= 1 - 4i - 3 + 4i + 2 \\ &= 0 \end{aligned}$$

Thus $z = i$ is a root.

$z = i$ is a root, hence the complex conjugate $z = -i$ is also a root.

First find the factor corresponding to each root.

Remember that the factor corresponding to the root $z = \alpha$ is $(z - \alpha)$.

root $z = i \Rightarrow$ factor $(z - i)$

root $z = -i \Rightarrow$ factor $(z - (-i))$

These two factors can be multiplied together to give a quadratic factor as follows.

$$(z - i)(z - (-i)) = (z - i)(z + i) = z^2 - i^2 = z^2 + 1 \quad [\text{since } i^2 = -1]$$

The other factor can be found using algebraic long division.

$$\begin{array}{r} z^2 + 4z + 2 \\ z^2 + 1 \overline{) z^4 + 4z^3 + 3z^2 + 4z + 2} \\ \underline{z^4 \qquad + z^2} \\ 4z^3 + 2z^2 + 4z + 2 \\ \underline{4z^3 \qquad + 4z} \\ 2z^2 + 2 \\ \underline{2z^2 + 2} \\ 0 \end{array}$$

$$\text{Hence } (z^2 + 1)(z^2 + 4z + 2) = 0$$

and the remaining roots come from the equation $z^2 + 4z + 2 = 0$.

$$a = 1, b = 4, c = 2 \Rightarrow b^2 - 4ac = 4^2 - 4(1)(2) = 8$$

$b^2 - 4ac > 0$, so the equation has two real and distinct roots which can be found using the quadratic formula.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{8}}{2(1)} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

The roots are $z = i$, $z = -i$, $z = -2 + \sqrt{2}$ and $z = -2 - \sqrt{2}$.

9. $z = -2 + i$ is given to be a root, hence the complex conjugate $z = -2 - i$ is also a root.

First find the factor corresponding to each root.

Remember that the factor corresponding to the root $z = \alpha$ is $(z - \alpha)$.

$$\text{root } z = -2 + i \Rightarrow \text{factor } (z - (-2 + i))$$

$$\text{root } z = -2 - i \Rightarrow \text{factor } (z - (-2 - i))$$

These two factors can be multiplied together to give a quadratic factor as follows.

$$\begin{aligned} (z - (-2 + i))(z - (-2 - i)) &= ((z + 2) - i)((z + 2) + i) \\ &= (z + 2)^2 - i^2 \\ &= z^2 + 4z + 4 + 1 \quad [\text{since } i^2 = -1] \\ &= z^2 + 4z + 5 \end{aligned}$$

The other factor can be found using algebraic long division.

$$\begin{array}{r} z^2 - 3z + 6 \\ z^2 + 4z + 5 \overline{) z^4 + z^3 - z^2 + 9z + 30} \\ \underline{z^4 + 4z^3 + 5z^2} \\ -3z^3 - 6z^2 + 9z + 30 \\ \underline{-3z^3 - 12z^2 - 15z} \\ 6z^2 + 24z + 30 \\ \underline{6z^2 + 24z + 30} \\ 0 \end{array}$$

$$\text{Hence } (z^2 + 4z + 5)(z^2 - 3z + 6) = 0$$

and the remaining roots come from the equation $z^2 - 3z + 6 = 0$.

$$a = 1, b = -3, c = 6 \Rightarrow b^2 - 4ac = (-3)^2 - 4(1)(6) = -15$$

$b^2 - 4ac < 0$, so the equation has non-real roots which can be found using the quadratic formula.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{-15}}{2(1)} = \frac{3 \pm \sqrt{15}i}{2}$$

The other roots are $z = -2 - i$, $z = \frac{3 + \sqrt{15}i}{2}$ and $z = \frac{3 - \sqrt{15}i}{2}$.

10.(a) $z = 2 + i$ is given to be a root, hence the complex conjugate $z = 2 - i$ is also a root.

(b) **Method 1**

$z = 2 + i$ is a root, so $z = 2 + i$ will satisfy the equation $z^4 - 6z^3 + 16z^2 - 22z + q = 0$.

Let $z = 2 + i$.

$$z^2 = (2 + i)^2 = 4 + 4i + i^2 = 4 + 4i - 1 = 3 + 4i$$

$$z^3 = zz^2 = (2 + i)(3 + 4i) = 6 + 11i + 4i^2 = 6 + 11i - 4 = 2 + 11i$$

$$z^4 = z^2 z^2 = (3 + 4i)(3 + 4i) = 9 + 24i + 16i^2 = 9 + 24i - 16 = -7 + 24i$$

$$z^4 - 6z^3 + 16z^2 - 22z + q = 0$$

$$\Rightarrow -7 + 24i - 6(2 + 11i) + 16(3 + 4i) - 22(2 + i) + q = 0$$

$$\Rightarrow -7 + 24i - 12 - 66i + 48 + 64i - 44 - 22i + q = 0$$

$$\Rightarrow -7 + 24i - 12 - 66i + 48 + 64i - 44 - 22i + q = 0$$

$$\Rightarrow -15 + q = 0$$

$$\Rightarrow q = 15$$

Hence the polynomial equation is $z^4 - 6z^3 + 16z^2 - 22z + 15 = 0$.

Now find the factor corresponding to each root.

Remember that the factor corresponding to the root $z = \alpha$ is $(z - \alpha)$.

$$\text{root } z = 2 + i \Rightarrow \text{factor } (z - (2 + i))$$

$$\text{root } z = 2 - i \Rightarrow \text{factor } (z - (2 - i))$$

These two factors can be multiplied together to give a quadratic factor as follows.

$$\begin{aligned} (z - (2 + i))(z - (2 - i)) &= ((z - 2) - i)((z - 2) + i) \\ &= (z - 2)^2 - i^2 \\ &= z^2 - 4z + 4 + 1 \quad [\text{since } i^2 = -1] \\ &= z^2 - 4z + 5 \end{aligned}$$

The other factor can be found using algebraic long division.

$$\begin{array}{r}
 z^2 - 4z + 5 \overline{) z^4 - 6z^3 + 16z^2 - 22z + 15} \\
 \underline{z^4 - 4z^3 + 5z^2} \\
 -2z^3 + 11z^2 - 22z + 15 \\
 \underline{-2z^3 + 8z^2 - 10z} \\
 3z^2 - 12z + 15 \\
 \underline{3z^2 - 12z + 15} \\
 0
 \end{array}$$

Hence $(z^2 - 4z + 5)(z^2 - 2z + 3) = 0$

and the remaining roots come from the equation $z^2 - 2z + 3 = 0$.

$$a = 1, b = -2, c = 3 \Rightarrow b^2 - 4ac = (-2)^2 - 4(1)(3) = -8$$

$b^2 - 4ac < 0$, so the equation has non-real roots which can be found using the quadratic formula.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{-8}}{2(1)} = \frac{2 \pm \sqrt{8}i}{2} = \frac{2 \pm 2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i$$

The remaining roots are $z = 1 + \sqrt{2}i$ and $z = 1 - \sqrt{2}i$.

Method 2

We know that $z = 2 + i$ and $z = 2 - i$ are roots.

First find the factor corresponding to each root.

Remember that the factor corresponding to the root $z = \alpha$ is $(z - \alpha)$.

root $z = 2 + i \Rightarrow$ factor $(z - (2 + i))$

root $z = 2 - i \Rightarrow$ factor $(z - (2 - i))$

These two factors can be multiplied together to give a quadratic factor as follows.

$$\begin{aligned}
 (z - (2 + i))(z - (2 - i)) &= ((z - 2) - i)((z - 2) + i) \\
 &= (z - 2)^2 - i^2 \\
 &= z^2 - 4z + 4 + 1 \quad [\text{since } i^2 = -1] \\
 &= z^2 - 4z + 5
 \end{aligned}$$

The value of q and the other factor can be found using algebraic long division.

$$\begin{array}{r}
 z^2 - 2z + 3 \\
 \hline
 z^2 - 4z + 5 \overline{) z^4 - 6z^3 + 16z^2 - 22z + q} \\
 \underline{z^4 - 4z^3 + 5z^2} \\
 -2z^3 + 11z^2 - 22z + q \\
 \underline{-2z^3 + 8z^2 - 10z} \\
 3z^2 - 12z + q \\
 \underline{3z^2 - 12z + 15} \\
 q - 15
 \end{array}$$

Since $z^2 - 4z + 5$ is a factor of the polynomial, the remainder must equal zero.

$$\text{Hence } q - 15 = 0 \Rightarrow q = 15$$

The remaining roots come from the equation $z^2 - 2z + 3 = 0$.

$$a = 1, b = -2, c = 3 \Rightarrow b^2 - 4ac = (-2)^2 - 4(1)(3) = -8$$

$b^2 - 4ac < 0$, so the equation has non-real roots which can be found using the quadratic formula.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{-8}}{2(1)} = \frac{2 \pm \sqrt{8}i}{2} = \frac{2 \pm 2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i$$

The remaining roots are $z = 1 + \sqrt{2}i$ and $z = 1 - \sqrt{2}i$.