ADVANCED HIGHER MATHEMATICS

## Solutions to Exam Questions on the Binomial Theorem

**1.** $(a^2-3)^4$

$$=\binom{4}{0}(a^2)^4(-3)^0+\binom{4}{1}(a^2)^3(-3)^1+\binom{4}{2}(a^2)^2(-3)^2+\binom{4}{3}(a^2)^1(-3)^3+\binom{4}{4}(a^2)^0(-3)^4$$

$$=a^8+4a^6(-3)+6a^4(9)+4a^2(-27)+81$$

$$=a^8-12a^6+54a^4-108a^2+81$$

**2.** $\left(\frac{1}{2}x-3\right)^4$

$$=\binom{4}{0}\left(\frac{1}{2}x\right)^4(-3)^0+\binom{4}{1}\left(\frac{1}{2}x\right)^3(-3)^1+\binom{4}{2}\left(\frac{1}{2}x\right)^2(-3)^2+\binom{4}{3}\left(\frac{1}{2}x\right)^1(-3)^3+\binom{4}{4}\left(\frac{1}{2}x\right)^0(-3)^4$$

$$=\frac{1}{16}x^4+4\left(\frac{1}{8}x^3\right)(-3)+6\left(\frac{1}{4}x^2\right)(9)+4\left(\frac{1}{2}x\right)(-27)+81$$

$$=\frac{1}{16}x^4-\frac{3}{2}x^3+\frac{27}{2}x^2-54x+81$$

**3.** $\left(x-\frac{2}{x}\right)^4=(x-2x^{-1})^4$

$$=\binom{4}{0}x^4(-2x^{-1})^0+\binom{4}{1}x^3(-2x^{-1})^1+\binom{4}{2}x^2(-2x^{-1})^2+\binom{4}{3}x^1(-2x^{-1})^3+\binom{4}{4}x^0(-2x^{-1})^4$$

$$=x^4+4x^3(-2x^{-1})+6x^2(4x^{-2})+4x(-8x^{-3})+16x^{-4}$$

$$=x^4-8x^2+24x^0-32x^{-2}+16x^{-4}$$

$$=x^4-8x^2+24-\frac{32}{x^2}+\frac{16}{x^4}$$

**4.** $\left(b-\frac{2}{b}\right)^5=(b-2b^{-1})^5$

$$=\binom{5}{0}b^5(-2b^{-1})^0+\binom{5}{1}b^4(-2b^{-1})^1+\binom{5}{2}b^3(-2b^{-1})^2+\binom{5}{3}b^2(-2b^{-1})^3+\binom{5}{4}b^1(-2b^{-1})^4+\binom{5}{5}b^0(-2b^{-1})^5$$

$$=b^5+5b^4(-2b^{-1})+10b^3(4b^{-2})+10b^2(-8b^{-3})+5b(16b^{-4})-32b^{-5}$$

$$=b^5-10b^3+40b-80b^{-1}+80b^{-3}-32b^{-5}$$

$$=b^5-10b^3+40b-\frac{80}{b}+\frac{80}{b^3}-\frac{32}{b^5}$$

**5.** $\left(x^2-\frac{2}{x}\right)^4=(x^2-2x^{-1})^4$

$$=\binom{4}{0}(x^2)^4(-2x^{-1})^0+\binom{4}{1}(x^2)^3(-2x^{-1})^1+\binom{4}{2}(x^2)^2(-2x^{-1})^2+\binom{4}{3}(x^2)^1(-2x^{-1})^3+\binom{4}{4}(x^2)^0(-2x^{-1})^4$$

$$=x^8+4x^6(-2x^{-1})+6x^4(4x^{-2})+4x^2(-8x^{-3})+16x^{-4}$$

$$=x^8-8x^5+24x^2-32x^{-1}+16x^{-4}$$

$$=x^8-8x^5+24x^2-\frac{32}{x}+\frac{16}{x^4}$$

**6.** $\left(x^3-\frac{2}{x}\right)^4=(x^3-2x^{-1})^4$

$$=\binom{4}{0}(x^3)^4(-2x^{-1})^0+\binom{4}{1}(x^3)^3(-2x^{-1})^1+\binom{4}{2}(x^3)^2(-2x^{-1})^2+\binom{4}{3}(x^3)^1(-2x^{-1})^3+\binom{4}{4}(x^3)^0(-2x^{-1})^4$$

$$=x^{12}+4x^9(-2x^{-1})+6x^6(4x^{-2})+4x^3(-8x^{-3})+16x^{-4}$$

$$=x^{12}-8x^8+24x^4-32x^0+16x^{-4}$$

$$=x^{12}-8x^8+24x^4-32+\frac{16}{x^4}$$

**7.** $\left(2a-\frac{3}{a}\right)^4=(2a-3a^{-1})^4$

$$=\binom{4}{0}(2a)^4(-3a^{-1})^0+\binom{4}{1}(2a)^3(-3a^{-1})^1+\binom{4}{2}(2a)^2(-3a^{-1})^2+\binom{4}{3}(2a)^1(-3a^{-1})^3+\binom{4}{4}(2a)^0(-3a^{-1})^4$$

$$=16a^4+4(8a^3)(-3a^{-1})+6(4a^2)(9a^{-2})+4(2a)(-27a^{-3})+81a^{-4}$$

$$=16a^4-96a^2+216a^0-216a^{-2}+81a^{-4}$$

$$=16a^4-96a^2+216-\frac{216}{a^2}+\frac{81}{a^4}$$

**8.** $\left(3x-\frac{2}{x^2}\right)^4=(3x-2x^{-2})^4$

$$=\binom{4}{0}(3x)^4(-2x^{-2})^0+\binom{4}{1}(3x)^3(-2x^{-2})^1+\binom{4}{2}(3x)^2(-2x^{-2})^2+\binom{4}{3}(3x)^1(-2x^{-2})^3+\binom{4}{4}(3x)^0(-2x^{-2})^4$$

$$=81x^4+4(27x^3)(-2x^{-2})+6(9x^2)(4x^{-4})+4(3x)(-8x^{-6})+16x^{-8}$$

$$=81x^4-216x+216x^{-2}-96x^{-5}+16x^{-8}$$

$$=81x^4-216x+\frac{216}{x^2}-\frac{96}{x^5}+\frac{16}{x^8}$$

**9.** $\left(\dfrac{2}{y^2}-5y\right)^3=(2y^{-2}-5y)^3$

$$=\binom{3}{0}(2y^{-2})^3(-5y)^0+\binom{3}{1}(2y^{-2})^2(-5y)^1+\binom{3}{2}(2y^{-2})^1(-5y)^2+\binom{3}{3}(2y^{-2})^0(-5y)^3$$

$$=8y^{-6}+3(4y^{-4})(-5y)+3(2y^{-2})(25y^2)-125y^3$$

$$=8y^{-6}-60y^{-3}+150y^0-125y^3$$

$$=\frac{8}{y^6}-\frac{60}{y^3}+150-125y^3$$

**10.** $\left(\dfrac{x^2}{3}-\dfrac{2}{x}\right)^5=\left(\dfrac{1}{3}x^2-2x^{-1}\right)^5$

$$=\binom{5}{0}\left(\frac{1}{3}x^2\right)^5(-2x^{-1})^0+\binom{5}{1}\left(\frac{1}{3}x^2\right)^4(-2x^{-1})^1+\binom{5}{2}\left(\frac{1}{3}x^2\right)^3(-2x^{-1})^2+\binom{5}{3}\left(\frac{1}{3}x^2\right)^2(-2x^{-1})^3$$

$$+\binom{5}{4}\left(\frac{1}{3}x^2\right)^1(-2x^{-1})^4+\binom{5}{5}\left(\frac{1}{3}x^2\right)^0(-2x^{-1})^5$$

$$=\frac{1}{243}x^{10}+5\left(\frac{1}{81}x^8\right)(-2x^{-1})+10\left(\frac{1}{27}x^6\right)(4x^{-2})+10\left(\frac{1}{9}x^4\right)(-8x^{-3})+5\left(\frac{1}{3}x^2\right)(16x^{-4})-32x^{-5}$$

$$=\frac{1}{243}x^{10}-\frac{10}{81}x^7+\frac{40}{27}x^4-\frac{80}{9}x+\frac{80}{3}x^{-2}-32x^{-5}$$

$$=\frac{x^{10}}{243}-\frac{10x^7}{81}+\frac{40x^4}{27}-\frac{80x}{9}+\frac{80}{3x^2}-\frac{32}{x^5}$$

**11.**(a) $(e^x+2)^4=\binom{4}{0}(e^x)^4 2^0+\binom{4}{1}(e^x)^3 2^1+\binom{4}{2}(e^x)^2 2^2+\binom{4}{3}(e^x)^1 2^3+\binom{4}{4}(e^x)^0 2^4$

$$=e^{4x}+4(e^{3x})2+6(e^{2x})4+4(e^x)8+16$$

$$=e^{4x}+8e^{3x}+24e^{2x}+32e^x+16$$

(b) $\int(e^x+2)^4\,dx=\int(e^{4x}+8e^{3x}+24e^{2x}+32e^x+16)dx$

$$=\frac{1}{4}e^{4x}+8\left(\frac{1}{3}e^{3x}\right)+24\left(\frac{1}{2}e^{2x}\right)+32e^x+16x+C$$

$$=\frac{1}{4}e^{4x}+\frac{8}{3}e^{3x}+12e^{2x}+32e^x+16x+C$$

**12.** $(x^2 + 3x)^8$

general term $= \binom{n}{r} a^{n-r} b^r$ where $a = x^2$, $b = 3x$ and $n = 8$

$= \binom{8}{r} (x^2)^{8-r} (3x)^r$

$= \binom{8}{r} x^{16-2r} 3^r x^r$

$= \binom{8}{r} 3^r x^{16-2r+r}$

$= \binom{8}{r} 3^r x^{16-r}$ where $0 \le r \le 8$

The term in $x^{13}$ occurs when $16 - r = 13 \Rightarrow r = 3$

Term in $x^{13} = \binom{8}{3} 3^3 x^{16-3} = 56(27)x^{13} = 1512x^{13}$

Hence the coefficient of $x^{13}$ is 1512.

**13.** $\left(x^2 + \frac{1}{x}\right)^{10} = (x^2 + x^{-1})^{10}$

general term $= \binom{n}{r} a^{n-r} b^r$ where $a = x^2$, $b = x^{-1}$ and $n = 10$

$= \binom{10}{r} (x^2)^{10-r} (x^{-1})^r$

$= \binom{10}{r} x^{20-2r} x^{-r}$

$= \binom{10}{r} x^{20-2r-r}$

$= \binom{10}{r} x^{20-3r}$ where $0 \le r \le 10$

The term in $x^{14}$ occurs when $20 - 3r = 14 \Rightarrow 3r = 6 \Rightarrow r = 2$

Term in $x^{14} = \binom{10}{2} x^{20-3(2)} = 45x^{14}$

**14.**(a) $\left(2x+\frac{5}{x^2}\right)^9 = (2x+5x^{-2})^9$

general term $= \binom{n}{r}a^{n-r}b^r$ where $a = 2x$, $b = 5x^{-2}$ and $n = 9$

$$= \binom{9}{r}(2x)^{9-r}(5x^{-2})^r$$

$$= \binom{9}{r}2^{9-r}x^{9-r}5^r x^{-2r}$$

$$= \binom{9}{r}2^{9-r}5^r x^{9-r-2r}$$

$$= \binom{9}{r}2^{9-r}5^r x^{9-3r} \text{ where } 0 \le r \le 9$$

(b) The term in independent of *x* is the term in $x^0$.

The term independent of *x* occurs when $9-3r=0 \Rightarrow 3r=9 \Rightarrow r=3$

Term independent of $x = \binom{9}{3}2^{9-3}5^3 x^{9-3(3)} = 84(64)(125)x^0 = 672\ 000$

**15.** $\left(\frac{1}{x}+3x\right)^{10} = (x^{-1}+3x)^{10}$

general term $= \binom{n}{r}a^{n-r}b^r$ where $a = x^{-1}$, $b = 3x$ and $n = 10$

$$= \binom{10}{r}(x^{-1})^{10-r}(3x)^r$$

$$= \binom{10}{r}x^{-10+r}3^r x^r$$

$$= \binom{10}{r}3^r x^{-10+r+r}$$

$$= \binom{10}{r}3^r x^{2r-10} \text{ where } 0 \le r \le 10$$

The term in $x^6$ occurs when $2r-10=6 \Rightarrow 2r=16 \Rightarrow r=8$

Term in $x^6 = \binom{10}{8}3^8 x^{2(8)-10} = 45(6561)x^6 = 295\ 245\,x^6$

**16.** $\left(\frac{3}{x}-2x\right)^{13} = (3x^{-1}-2x)^{13}$

$$\text{general term} = \binom{n}{r}a^{n-r}b^{r} \qquad \text{where } a = 3x^{-1},\ b=-2x \text{ and } n=13$$

$$= \binom{13}{r}(3x^{-1})^{13-r}(-2x)^{r}$$

$$= \binom{13}{r}3^{13-r}x^{-13+r}(-2)^{r}x^{r}$$

$$= \binom{13}{r}3^{13-r}(-2)^{r}x^{-13+r+r}$$

$$= \binom{13}{r}3^{13-r}(-2)^{r}x^{2r-13} \qquad \text{where } 0 \le r \le 13$$

The term in $x^9$ occurs when $2r-13=9 \quad \Rightarrow \quad 2r=22 \quad \Rightarrow \quad r=11$

Term in $x^9 = \binom{13}{11}3^{13-11}(-2)^{11}x^{2(11)-13} = 78(9)(-2048)x^9 = -1\,437\,696\,x^9$

**17.** $\left(2x-\frac{1}{x^2}\right)^{9} = (2x-x^{-2})^{9}$

$$\text{general term} = \binom{n}{r}a^{n-r}b^{r} \qquad \text{where } a = 2x,\ b=-x^{-2} \text{ and } n=9$$

$$= \binom{9}{r}(2x)^{9-r}(-x^{-2})^{r}$$

$$= \binom{9}{r}2^{9-r}x^{9-r}(-1)^{r}x^{-2r}$$

$$= \binom{9}{r}2^{9-r}(-1)^{r}x^{9-r-2r}$$

$$= \binom{9}{r}2^{9-r}(-1)^{r}x^{9-3r} \qquad \text{where } 0 \le r \le 9$$

The term in independent of *x* is the term in $x^0$.

The term independent of *x* occurs when $9-3r=0 \quad \Rightarrow \quad 3r=9 \quad \Rightarrow \quad r=3$

Term independent of $x = \binom{9}{3}2^{9-3}(-1)^{3}x^{9-3(3)} = 84(64)(-1)x^0 = -5376$

**18.** $\left(2x - \frac{5}{x^2}\right)^6 = (2x - 5x^{-2})^6$

general term $= \binom{n}{r} a^{n-r} b^r$ where $a = 2x$, $b = -5x^{-2}$ and $n = 6$

$$= \binom{6}{r} (2x)^{6-r} (-5x^{-2})^r$$

$$= \binom{6}{r} 2^{6-r} x^{6-r} (-5)^r x^{-2r}$$

$$= \binom{6}{r} 2^{6-r} (-5)^r x^{6-r-2r}$$

$$= \binom{6}{r} 2^{6-r} (-5)^r x^{6-3r} \quad \text{where } 0 \le r \le 6$$

The term in independent of $x$ is the term in $x^0$.

The term independent of $x$ occurs when $6 - 3r = 0 \Rightarrow 3r = 6 \Rightarrow r = 2$

Term independent of $x = \binom{6}{2} 2^{6-2} (-5)^2 x^{6-3(2)} = 15(16)(25)x^0 = 600$

**19.** $\left(\frac{2}{x}+\frac{1}{4x^2}\right)^{10}=\left(2x^{-1}+\frac{1}{4}x^{-2}\right)^{10}$

general term $=\binom{n}{r}a^{n-r}b^r$ where $a=2x^{-1}$, $b=\frac{1}{4}x^{-2}$ and $n=10$

$$=\binom{10}{r}(2x^{-1})^{10-r}\left(\frac{1}{4}x^{-2}\right)^r$$

$$=\binom{10}{r}2^{10-r}x^{-10+r}\left(\frac{1}{4}\right)^r x^{-2r}$$

$$=\binom{10}{r}2^{10-r}x^{-10+r}(4^{-1})^r x^{-2r}$$

$$=\binom{10}{r}2^{10-r}x^{-10+r}4^{-r}x^{-2r}$$

$$=\binom{10}{r}2^{10-r}x^{-10+r}(2^2)^{-r}x^{-2r}$$

$$=\binom{10}{r}2^{10-r}x^{-10+r}2^{-2r}x^{-2r}$$

$$=\binom{10}{r}2^{10-r-2r}x^{-10+r-2r}$$

$$=\binom{10}{r}2^{10-3r}x^{-10-r} \qquad \text{where } 0\le r\le 10$$

The term in $\frac{1}{x^{13}}$ is the term in $x^{-13}$.

The term in $\frac{1}{x^{13}}$ occurs when $-10-r=-13 \quad\Rightarrow\quad r=3$

Term in $\frac{1}{x^{13}}=\binom{10}{3}2^{10-3(3)}x^{-10-3}=120(2)x^{-13}=240x^{-13}=240\left(\frac{1}{x^{13}}\right)$ *or* $\frac{240}{x^{13}}$

**<u>Note</u>**

In order to fully simplify the expression for the general term, you must write 4 as $2^2$ and combine the powers of 2.

**20.** We must prove that the LHS of the identity is equal to the RHS.

First simplify each of the binomial coefficients on the LHS using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

$$\binom{n+2}{3} = \frac{(n+2)!}{3!\left((n+2)-3\right)!} = \frac{(n+2)!}{6(n-1)!}$$

$$= \frac{(n+2)(n+1)n(n-1)...3.2.1}{6(n-1)(n-2)(n-3)...3.2.1}$$

$$= \frac{(n+2)(n+1)n}{6} \qquad \text{[cancelling } (n-1)(n-2)(n-3)...3.2.1\text{]}$$

$$= \frac{n(n+1)(n+2)}{6}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n!}{6(n-3)!}$$

$$= \frac{n(n-1)(n-2)(n-3)...3.2.1}{6(n-3)(n-4)(n-5)...3.2.1}$$

$$= \frac{n(n-1)(n-2)}{6} \qquad \text{[cancelling } (n-3)(n-4)(n-5)...3.2.1\text{]}$$

$$\text{LHS} = \binom{n+2}{3} - \binom{n}{3} = \frac{n(n+1)(n+2)}{6} - \frac{n(n-1)(n-2)}{6}$$

$$= \frac{n(n+1)(n+2) - n(n-1)(n-2)}{6}$$

$$= \frac{n(n^2+3n+2) - n(n^2-3n+2)}{6}$$

$$= \frac{n^3+3n^2+2n-n^3+3n^2-2n}{6}$$

$$= \frac{6n^2}{6}$$

$$= n^2$$

$$= \text{RHS}$$

LHS = RHS, hence $\binom{n+2}{3} - \binom{n}{3} = n^2$.

**21.** We must prove that the LHS of the identity is equal to the RHS.

First simplify each of the binomial coefficients on the LHS using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

$$\binom{n+1}{3} = \frac{(n+1)!}{3!\left((n+1)-3\right)!} = \frac{(n+1)!}{6(n-2)!}$$

$$= \frac{(n+1)n(n-1)(n-2)...3.2.1}{6(n-2)(n-3)(n-4)...3.2.1}$$

$$= \frac{(n+1)n(n-1)}{6}$$ [cancelling $(n-2)(n-3)(n-4)...3.2.1$]

$$= \frac{n(n+1)(n-1)}{6}$$

$$\binom{n}{3} = \frac{n!}{3!\left(n-3\right)!} = \frac{n!}{6(n-3)!}$$

$$= \frac{n(n-1)(n-2)(n-3)...3.2.1}{6(n-3)(n-4)(n-5)...3.2.1}$$

$$= \frac{n(n-1)(n-2)}{6}$$ [cancelling $(n-3)(n-4)(n-5)...3.2.1$]

$$\text{LHS} = \binom{n+1}{3} - \binom{n}{3} = \frac{n(n+1)(n-1)}{6} - \frac{n(n-1)(n-2)}{6}$$

$$= \frac{n(n+1)(n-1) - n(n-1)(n-2)}{6}$$

$$= \frac{n(n^2-1) - n(n^2-3n+2)}{6}$$

$$= \frac{n^3 - n - n^3 + 3n^2 - 2n}{6}$$

$$= \frac{3n^2 - 3n}{6}$$ [divide all terms in the fraction by 3 to simplify]

$$= \frac{n^2 - n}{2}$$

$$= \frac{n(n-1)}{2}$$

Now simplify the binomial coefficient on the RHS.

$$\text{RHS} = \binom{n}{2} = \frac{n!}{2!\left(n-2\right)!} = \frac{n!}{2(n-2)!}$$

$$= \frac{n(n-1)(n-2)...3.2.1}{2(n-2)(n-3)(n-4)...3.2.1}$$

$$= \frac{n(n-1)}{2}$$ [cancelling $(n-2)(n-3)(n-4)...3.2.1$]

LHS = RHS, hence $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$.