Higher Maths 2006

Paper 2 Solutions

1. (a)

$$m_{PS} = \frac{6-0}{4-2}$$

$$m_{PS} = \frac{6}{2}$$

$$m_{PS} = 3$$

For perpendicular lines,
$m_1 \cdot m_2 = -1 \Rightarrow m_{QS} = -\frac{1}{3}$

$$y - 6 = -\tfrac{1}{3}(x-4)$$

$$3y - 18 = -x + 4$$

$$x + 3y = 22$$

Equation of QS is $x + 3y = 22$

(b) QS crosses x-axis when $y = 0$,

$\Rightarrow \quad x = 22$

Q(22,0)

$\Rightarrow \quad PQ = 22 - 2$

$= 20$ units

$\Rightarrow \quad R(24,6)$

2.

$$kx^2 + kx + 6 = 0$$

For equal roots, $b^2 - 4ac = 0$

$a = k \qquad b = k \qquad c = 6$

$$(k)^2 - 4(k)(6) = 0$$

$$k^2 - 24k = 0$$

$$k(k-24) = 0$$

$\Rightarrow \quad k = 0, 24$

For equal roots $k = 24$ as $k \neq 0$.

3. (a)

$$y = x^2 - 14x + 53$$
$$\frac{dy}{dx} = 2x - 14$$

when $x = 8$, $m = 2(8) - 14$
$m = 2$

$y - 5 = 2(x - 8)$
$y - 5 = 2x - 16$
$y = 2x - 11$

(b) For points of intersection, $y = y$:

$$2x - 11 = -x^2 + 10x - 27$$
$$x^2 - 8x + 16 = 0$$
$$(x-4)(x-4) = 0$$
$$x = 4 \quad \text{twice}$$

Single point of intersection $\Rightarrow$ tangency

or check discriminant:

$$b^2 - 4ac = (-8)^2 - 4(1)(16)$$
$$= 64 - 64$$
$$= 0$$
$$\Rightarrow \text{tangency}$$

when $x = 4$, $y = 2(4) - 11$
$y = -3$

$\Rightarrow$ Q(4,–3)

4. $(x-3)^2 + (y-4)^2 = 25$

C(3,4)

$x^2 + y^2 - kx - 8y - 2k = 0$

$\Rightarrow$ $k = 6$

$x^2 + y^2 - 6x - 8y - 12 = 0$

$r = \sqrt{9 + 16 - (-12)}$

$r = \sqrt{37}$

The larger circle has radius $\sqrt{37}$.

5. $\frac{dy}{dx} = 4x - 6x^2 \qquad (-1,9)$

$y = \int\left(4x - 6x^2\right)dx$

$y = \frac{4x^2}{2} - \frac{6x^3}{3} + C$

$y = 2x^2 - 2x^3 + C$

when $x = -1$, $y = 9$

$\Rightarrow \quad 9 = 2(-1)^2 - 2(-1)^3 + C$

$9 = 2 - 2 + C$

$9 = 4 + C$

$5 = C$

$\Rightarrow \quad y = 2x^2 - 2x^3 + 5$

6. (a) $P(-1,2,-1) \qquad Q(3,2,-4)$

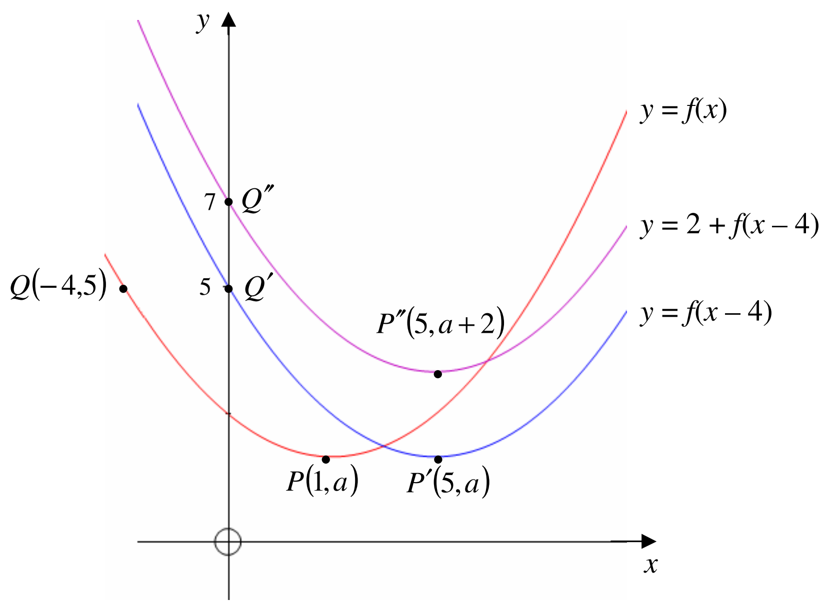
$$\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

(b)
$$\left|\overrightarrow{PQ}\right| = \sqrt{(4)^2 + (0)^2 + (-3)^2}$$
$$= \sqrt{16+9}$$
$$= \sqrt{25}$$
$$= 5$$

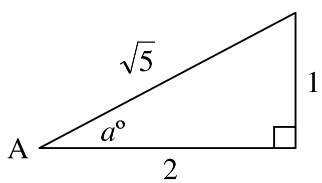
(c) $\left|\overrightarrow{PQ}\right| = 5 \quad \Rightarrow \quad \frac{1}{5}\left|\overrightarrow{PQ}\right| = 1$ (unit vector)

$\Rightarrow \quad \frac{1}{5}\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ 0 \\ -\frac{3}{5} \end{pmatrix}$ is a unit vector parallel to $\overrightarrow{PQ}$.

7. (a) (b)



8. (a)



(*i*) $\sin a^\circ = \frac{1}{\sqrt{5}}$

(*ii*) $\cos a^\circ = \frac{2}{\sqrt{5}}$

$$\begin{aligned}\sin 2a^\circ &= 2\sin a^\circ \cos a^\circ \\ &= 2\cdot\frac{1}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}} \\ &= \frac{4}{5}\end{aligned}$$

(b)

$$\begin{aligned}\sin 3a^\circ &= \sin(2a+a)^\circ \\ &= \sin 2a^\circ \cos a^\circ + \cos 2a^\circ \sin a^\circ \\ &= \frac{4}{5}\cdot\frac{2}{\sqrt{5}} + \frac{3}{5}\cdot\frac{1}{\sqrt{5}} \\ &= \frac{8}{5\sqrt{5}} + \frac{3}{5\sqrt{5}} \\ &= \frac{11}{5\sqrt{5}}\end{aligned}$$

$$\begin{aligned}\cos 2a^\circ &= 1-2\sin^2 a^\circ \\ &= 1-2\left(\frac{1}{\sqrt{5}}\right)^2 \\ &= 1-2\left(\frac{1}{5}\right) \\ &= 1-\frac{2}{5} \\ &= \frac{3}{5}\end{aligned}$$

9.

$$y = \frac{1}{x^3} - \cos 2x$$

$$y = x^{-3} - \cos 2x$$

$$\frac{dy}{dx} = -3x^{-4} - (-\sin 2x)\cdot 2$$

$$= \frac{-3}{x^4} + 2\sin 2x$$

$$= 2\sin 2x - \frac{3}{x^4}$$

10. (a)

$$y = 7\sin x - 24\cos x$$

$$k\sin(x-a) = k\sin x\cos a - k\cos x\sin a$$

Using Radian mode:

$k\cos a = 7$

$k\sin a = 24$

$$k = \sqrt{(7)^2 + (24)^2}$$

$$k = \sqrt{49+576}$$

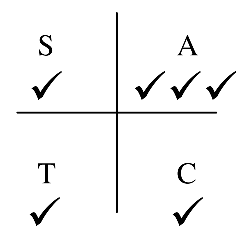
$$k = \sqrt{625}$$

$$k = 25$$

$$\tan a = \frac{k\sin a}{k\cos a}$$

$$\tan a = \frac{24}{7}$$

(acute angle) $a = 1.29$ radians *



a is in 1st quadrant $\Rightarrow a = 1.29$ radians

$$\Rightarrow y = 25\sin(x - 1.29)$$

* Working in degrees then converting to radians gives:

$$\tan a^\circ = \frac{24}{7}$$

$$a = 73.7^\circ$$

$$\Rightarrow a = 73.7 \cdot \frac{\pi}{180} \text{ radians}$$

$$a = 1.29 \text{ radians}$$

(b)

$$\frac{dy}{dx} = 25\cos(x-1.29)$$

$$\Rightarrow 25\cos(x-1.29) = 1$$

$$\cos(x-1.29) = \frac{1}{25}$$

$$x - 1.29 = \cos^{-1}\left(\frac{1}{25}\right)$$

$$x - 1.29 = 1.53$$

$$x = 2.82 \text{ radians}$$

11.

$$A(t) = A_o e^{-0.000124t}$$

$$0.88 = 1 \cdot e^{-0.000124t}$$

$$\log_e 0.88 = \log_e e^{-0.000124t}$$

$$\log_e 0.88 = -0.000124t \log_e e \qquad (\log_e e = 1)$$

$$\log_e 0.88 = -0.000124t$$

$$t = \frac{\log_e 0.88}{-0.000124}$$

t = 1031 years (to the nearest year)

$\Rightarrow$ The claim is true.

12. (a) (*i*) $PS = 6 - \mathrm{x}$

$$RS = 12 - \frac{8}{x}$$

(*ii*) $A = PS \cdot RS$

$$A = (6-x)\left(12 - \frac{8}{x}\right)$$

$$A = 72 - \frac{48}{x} - 12x + \frac{8x}{x}$$

$$A = 72 - \frac{48}{x} - 12x + 8$$

$$A = 80 - 12x - \frac{48}{x}$$

Proved

(b) P.T.O.

(b) $A = 80 - 12x - 48x^{-1}$

$$A' = -12 + 48x^{-2}$$

$$A' = \frac{48}{x^2} - 12$$

For stationary points $A' = 0$

$$\Rightarrow \quad \frac{48}{x^2} - 12 = 0$$

$$48 - 12x^2 = 0$$

$$4 - x^2 = 0$$

$$(2+x)(2-x) = 0$$

$$\Rightarrow \quad x = -2,\ 2$$

$$x = 2 \quad \text{as} \quad x > 0$$

when $x = 2$,

$$A = 80 - 12(2) - \frac{48}{(2)}$$

$$A = 80 - 24 - 24$$

$$A = 32$$

x	$\rightarrow$	2	$\rightarrow$
A'	+	0	–
Shape	/	—	\

$\Rightarrow$ Maximum turning point at $(2,32)$

$$A'(1) = \frac{48}{(1)^2} - 12$$

$$= 32$$

$$A'(3) = \frac{48}{(3)^2} - 12$$

$$= \frac{48}{9} - \frac{108}{9}$$

$$= -\frac{60}{9}$$

Check endpoints:

when $x = 1$;

$$A = 80 - 12(1) - \frac{48}{(1)}$$

$$A = 80 - 12 - 48$$

$$A = 20$$

when $x = 4$;

$$A = 80 - 12(4) - \frac{48}{(4)}$$

$$A = 80 - 48 - 12$$

$$A = 20$$

$\Rightarrow$ Maximum Area of 32 units^2 when $x = 2$

Minimum Area of 20 units^2 when $x = 1, 4$