Higher Maths 2006

Paper 1 Solutions

1. (a) Midpoint $D\left(\frac{-1+7}{2},\frac{12+(-2)}{2}\right) = (3,5)$

Using B(–2,–5) and D(3,5);

Gradient

$$m = \frac{5-(-5)}{3-(-2)}$$

$$m = \frac{10}{5}$$

$$m = 2$$

$$y - b = m(x - a)$$

$$y - (-5) = 2(x - (-2))$$

$$y + 5 = 2x + 4$$

$$y = 2x - 1$$

Equation of median BD is $y = 2x - 1$

(b) Gradient

$$m_{BC} = \frac{-2-(-5)}{7-(-2)}$$

$$m_{BC} = \frac{3}{9}$$

$$m_{BC} = \frac{1}{3}$$

For perpendicular lines, $m_1 \cdot m_2 = -1 \Rightarrow m_{AE} = -3$

$$y - b = m(x - a)$$

$$y - 12 = -3\,(x - (-1))$$

$$y - 12 = -3x - 3$$

$$y = -3x + 9$$

Equation of altitude AE is $y = -3x + 9$

(c) For points of intersection, $y = y$;

$$2x - 1 = -3x + 9$$

$$5x = 10$$

$$x = 2$$

when $x = 2$,

$$y = 2(2) - 1$$

$$y = 3$$

Point of intersection of BD and AE at (2,3)

2. (a) C(–2,3)

$$r^2 = (1-(-2))^2 + (6-3)^2$$

$$r^2 = (3)^2 + (3)^2$$

$$r^2 = 18$$

Equation of circle is $(x + 2)^2 + (y - 3)^2 = 18$

(b) At Q $y = 0$,

$$(x + 2)^2 + (0 - 3)^2 = 18$$

$$x^2 + 4x + 4 + 9 = 18$$

$$x^2 + 4x - 5 = 0$$

$$(x + 5)(x - 1) = 0$$

$\Rightarrow x = -5, 1$

$\Rightarrow$ Q(–5,0)

Gradient $m_r = \dfrac{3-0}{-2-(-5)}$

$m_r = \dfrac{3}{3}$

$m_r = 1$

For perpendicular lines, $m_1 \cdot m_2 = -1 \Rightarrow m_T = -1$

$$y - b = m(x - a)$$

$$y - 0 = -1\,(x - (-5))$$

$$y = -x - 5$$

Equation of tangent at Q is $x + y + 5 = 0$

3. (a) $f(x) = 2x + 3$ $g(x) = 2x - 3$

(i) $f(g(x)) = f(2x - 3)$

$= 2(2x - 3) + 3$

$= 4x - 6 + 3$

$= 4x - 3$

(ii) $g(f(x)) = g(2x + 3)$

$= 2(2x + 3) - 3$

$= 4x + 6 - 3$

$= 4x + 3$

(b) $f(g(x)) \times g(f(x)) = (4x - 3)(4x + 3)$

$= 16x^2 - 9$

$y = 16x^2 - 9$ is symmetrical about the y-axis.

When $x = 0$, $y = -9$.

Minimum value is -9.

4. (a) For a limit to exist $-1 < m < 1$.

$m = 0.8 \Rightarrow$ this sequence has a limit.

(b) $L = \dfrac{c}{1 - m}$

$L = \dfrac{12}{1 - 0.8}$

$L = \dfrac{12}{0.2}$

$L = 60$

5. $f(x)=(2x-1)^5$

$f'(x)=5(2x-1)^4\cdot 2$

$f'(x)=10(2x-1)^4$

For stationary points $f'(x)=0$.

$$\Rightarrow \quad 10(2x-1)^4 = 0$$
$$(2x-1)^4 = 0$$
$$2x-1 = 0$$
$$2x = 1$$
$$x = \frac{1}{2}$$

$$f\left(\tfrac{1}{2}\right)=\left(2\left(\tfrac{1}{2}\right)-1\right)^5$$
$$f\left(\tfrac{1}{2}\right)=(1-1)^5$$
$$f\left(\tfrac{1}{2}\right)=(0)^5$$
$$f\left(\tfrac{1}{2}\right)=0$$

$\Rightarrow$ Stationary point at $\left(\tfrac{1}{2},0\right)$

$$f'(0)=10(2(0)-1)^4$$
$$f'(0)=10(-1)^4$$
$$f'(0)=10$$

$$f'(1)=10(2(1)-1)^4$$
$$f'(1)=10(1)^4$$
$$f'(0)=10$$

x	$\rightarrow$	$\frac{1}{2}$	$\rightarrow$
$f'(x)$	+	0	+
Shape	/	—	/

$\Rightarrow$ Rising point of inflexion at $\left(\tfrac{1}{2},0\right)$

6. (a) $\int_0^1 \left(x^3 - 6x^2 + 4x + 1\right) dx$

$= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{4x^2}{2} + x\right]_0^1$

$= \left[\frac{1}{4}x^4 - 2x^3 + 2x^2 + x\right]_0^1$

$= \left(\frac{1}{4}(1)^4 - 2(1)^3 + 2(1)^2 + (1)\right) - \left(\frac{1}{4}(0)^4 - 2(0)^3 + 2(0)^2 + (0)\right)$

$= \frac{1}{4} - 2 + 2 + 1$

$= \frac{5}{4}$

$\Rightarrow$ Shaded area S $= \frac{5}{4}$ units^2

(b) $\int_1^2 \left(x^3 - 6x^2 + 4x + 1\right) dx$

$= \left[\frac{1}{4}x^4 - 2x^3 + 2x^2 + x\right]_1^2$

$= \left(\frac{1}{4}(2)^4 - 2(2)^3 + 2(2)^2 + (2)\right) - \left(\frac{1}{4}(1)^4 - 2(1)^3 + 2(1)^2 + (1)\right)$

$= \left(\frac{1}{4}(16) - 2(8) + 2(4) + 2\right) - \frac{5}{4}$

$= 4 - 16 + 8 + 2 - \frac{5}{4}$

$= -2 - \frac{5}{4}$

$= -\frac{13}{4}$

$\Rightarrow$ Total shaded area $= \frac{5}{4} + \frac{13}{4}$

$= \frac{18}{4}$

$= \frac{9}{2}$ units^2

7.

$$\sin x^\circ - \sin 2x^\circ = 0$$

$$\sin x^\circ - 2\sin x^\circ \cos x^\circ = 0$$

$$\sin x^\circ (1 - 2\cos x^\circ) = 0$$

$\Rightarrow \quad \sin x^\circ = 0$ or $1 - 2\cos x^\circ = 0$

$x^\circ = 0^\circ,\ 180^\circ,\ 360^\circ$ $\qquad \cos x^\circ = \frac{1}{2}$

$x^\circ = 60^\circ,\ 300^\circ$

$\Rightarrow \quad x \in \{0^\circ,\ 60^\circ,\ 180^\circ,\ 300^\circ,\ 360^\circ\}$ for $0 \le x \le 360$.

8. (a)

$$2x^2 + 4x - 3$$

$$= 2(x^2 + 2x) - 3$$

$$= 2[(x+1)^2 - 1] - 3$$

$$= 2(x+1)^2 - 2 - 3$$

$$= 2(x+1)^2 - 5$$

(b) Minimum turning point at $(-1, -5)$.

9. (a)

$$\begin{aligned} \boldsymbol{u}.\boldsymbol{v} &= 1 \\ (k^3)(1)+(1)(3k^2)+(k+2)(-1) &= 1 \\ k^3+3k^2-k-2 &= 1 \\ k^3+3k^2-k-3 &= 0 \end{aligned}$$

Proved

(b)

-3	1	3	-1	-3
		-3	0	3
	1	0	-1	0

$\Rightarrow (k+3)$ is a factor

Quotient = $(k-1)^2$

$$\begin{aligned} \Rightarrow \quad (k+3)(k-1)^2 &= 0 \\ (k+3)(k+1)(k-1) &= 0 \end{aligned}$$

(c) $\Rightarrow k = -3, -1, 1$

$\Rightarrow k = 1$ as $k > 0$

(d) When $k = 1$, $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

$$|\boldsymbol{u}| = \sqrt{1^2+1^2+3^2} = \sqrt{11}$$

$$|\boldsymbol{v}| = \sqrt{1^2+3^2+(-1)^2} = \sqrt{11}$$

$$\cos\theta = \frac{(1)(1)+(1)(3)+(3)(-1)}{\sqrt{11}\cdot\sqrt{11}}$$

$$\cos\theta = \frac{1}{11}$$

10.

$$y = a^x$$
$$\log_4 y = \log_4 a^x$$
$$\log_4 y = x\log_4 a$$
$$Y = mx$$

From graph, $m = \frac{3-0}{6-0}$

$$m = \frac{3}{6}$$
$$m = \frac{1}{2}$$

$\Rightarrow \quad \log_4 a = \frac{1}{2}$

$$a = 4^{\frac{1}{2}}$$
$$a = 2$$