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1) $\sin(p+q) = \sin p \cos q + \cos p \sin q$

$OA = \sqrt{6^2+8^2} = \sqrt{100} = \underline{\underline{10}}$

$OC = \sqrt{12^2+5^2} = \sqrt{169} = \underline{\underline{13}}$

$\sin p = \frac{8}{10} = \underline{\underline{\frac{4}{5}}}$ $\quad \cos p = \frac{6}{10} = \underline{\underline{\frac{3}{5}}}$

$\sin q = \underline{\underline{\frac{5}{13}}}$ $\quad \cos q = \underline{\underline{\frac{12}{13}}}$

$\therefore \sin(p+q) = \left(\frac{4}{5} \times \frac{12}{13}\right) + \left(\frac{3}{5} \times \frac{5}{13}\right)$

$= \frac{48}{65} + \frac{15}{65} = \underline{\underline{\frac{63}{65}}}$

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2) a) $f'(x) = 3x^2 - 12x + 9 = 0$ at T.P

$(3x-3)(x-3) = 0$

$x = \frac{3}{3} = 1$ OR $x = 3$

$3x \quad -3$
$x \quad -3$

$-9x$
$-3x$
$-12x$

$\therefore \underline{\underline{x=1}}$

at $x=1$: $y = 1^3 - (6\times1^2) + 9\times1$

$= 1 - 6 + 9 = \underline{\underline{4}}$

$\therefore$ T.P. at $\underline{\underline{(1,4)}}$

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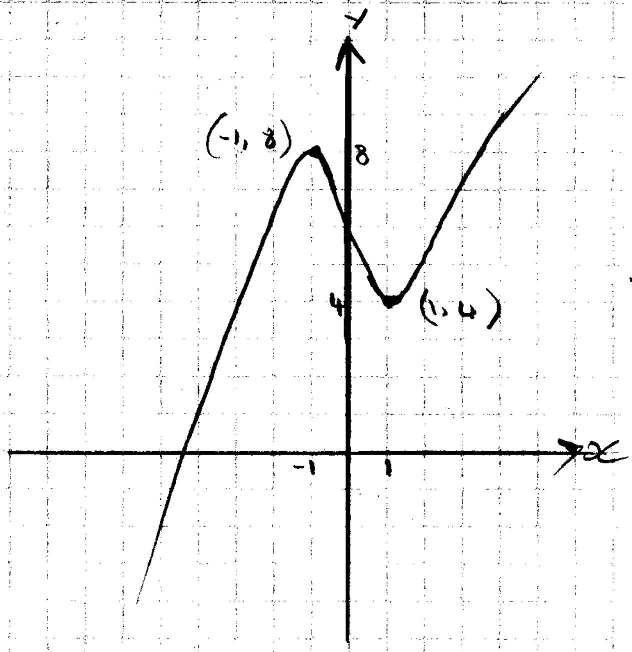
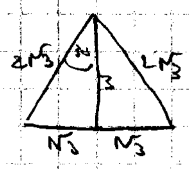
2) b)



c) $4 < k < 8$

3) $M = \tan\theta$

$$M = \frac{2 - -1}{3\sqrt{3} - 0} = \frac{3}{3\sqrt{3}} = \frac{\sqrt{3}}{3}$$



$$\therefore \tan\theta = \frac{\sqrt{3}}{3} \quad \therefore \theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$= 30^\circ$$

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4) $M = \frac{dy}{dx}$ $\quad M_A = 10x - 15$

$M_B = 3x^2 - 12$

at $M_A = M_B$: $3x^2 - 12 = 10x - 15$

$3x^2 - 10x + 3 = 0$

$(3x-1)(x-3) = 0$

$\therefore \underline{\underline{x = \frac{1}{3}}}$ OR $\underline{\underline{x = 3}}$

at $x = \frac{1}{3}$ the curves are parallel, at $x=3$ the curves touch and have a common tangent.

b) $$\int_{-1}^{3} (x^3 - 12x + 1) - (5x^2 - 15x - 8)\, dx$$

$$= \int_{-1}^{3} x^3 - 5x^2 + 3x + 9\, dx$$

$$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + \frac{3x^2}{2} + 9x \right]_{-1}^{3}$$

$$= \left(\frac{3^4}{4} - \frac{5\times3^3}{3} + \frac{3\times3^2}{2} + 9\times3 \right) - \left(\frac{-1^4}{4} - \frac{5\times-1^3}{3} + \frac{3\times-1^2}{2} + 9\times-1 \right)$$

$$= \left(\frac{81}{4} - \frac{135}{3} + \frac{27}{2} + 27 \right) - \left(\frac{1}{4} + \frac{5}{3} + \frac{3}{2} - 9 \right)$$

$$= \left(\frac{243}{12} - \frac{540}{12} + \frac{162}{12} + \frac{324}{12} \right) - \left(\frac{3}{12} + \frac{20}{12} + \frac{18}{12} - \frac{108}{12} \right)$$

$$= \frac{189}{12} + \frac{67}{12} = \frac{256}{12} = \underline{\underline{21\tfrac{1}{3}\text{ units}^2}}$$

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5) At the limit $L = \frac{b}{1-a}$

$\therefore U_{n+1} = aU_n + 10 \Rightarrow L = \frac{10}{1-a}$

$\therefore U_{n+1} = a^2U_n + 16 \Rightarrow L = \frac{16}{1-a^2}$

Now $L = L$: $\frac{10}{1-a} = \frac{16}{1-a^2}$

$10(1-a^2) = 16(1-a)$

$10 - 10a^2 = 16 - 16a$

$10a^2 - 16a + 6 = 0 \quad \div 2$

$5a^2 - 8a + 3 = 0$

$(5a-3)(a-1) = 0$

$\therefore a = \frac{3}{5}$ or $a = 1$ Now $-1 < a < 1$

$\therefore \underline{\underline{a = \frac{3}{5}}}$

$\therefore L = \frac{10}{1-\frac{3}{5}} = \frac{10}{\frac{2}{5}} = \frac{10 \times 5}{2} = \underline{\underline{25}}$

6) For a circle $r > 0$ $2g = 4k$ $2f = -2k$ $c = -k-2$

$\therefore r = \sqrt{g^2 + f^2 - c}$ $g = 2k$ $f = -k$

$r = \sqrt{4k^2 + k^2 + k + 2}$

$= \sqrt{5k^2 + k + 2}$

$\therefore 5k^2 + k + 2 > 0$

$\therefore$ All values of k

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7) $\vec{VK} = \vec{VA} + \vec{AB} + \vec{BK}$

If $\vec{AB} = \begin{pmatrix} 8 \\ -4 \\ 4 \end{pmatrix}$ then $\vec{BK} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\therefore \vec{VK} = \begin{pmatrix} -7 \\ -13 \\ -11 \end{pmatrix} + \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ -8 \\ -16 \end{pmatrix}}}$

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8) $f'(x) = \sin(3x)$ then

$y = \int \sin(3x) = -\frac{1}{3}\cos 3x + c$

at $\left(\frac{\pi}{9}, 1\right)$: $1 = -\frac{1}{3}\cos\frac{\pi}{3} + c$

$c = 1 + \frac{1}{3}\cos\frac{\pi}{3}$

$c = 1 + \frac{1}{3} \times \frac{1}{2} = 1 + \frac{1}{6} = \underline{\underline{\frac{7}{6}}}$

$\therefore \underline{\underline{y = -\frac{1}{3}\cos 3x + \frac{7}{6}}}$

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9) $\log_5 2 + \log_5 50 - \log_5 4$

$= \log_5 100 - \log_5 4$

$= \log_5 25$

$= \underline{\underline{2}}$

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10) $\cos x - \sin x = k\cos(x - \alpha)$

$= k(\cos x \cos\alpha - \sin x \sin\alpha)$

$\cos x - \sin x = k\cos x\cos\alpha + k\sin x\sin\alpha$

$k\cos\alpha = 1 \qquad k\sin\alpha = -1$

Squaring and adding: $k^2\cos^2\alpha + k^2\sin^2\alpha = 1^2 + -1^2$

$k^2(\cos^2\alpha + \sin^2\alpha) = 2$

$k^2 = 2$

$\underline{k = \sqrt{2}}$

NOW: $\dfrac{\cancel{k}\sin\alpha}{\cancel{k}\cos\alpha} = \dfrac{-1}{1} = \tan\alpha$

$\therefore \underline{\tan\alpha = -1}$

Sin is -ve, Cos α +ve $\therefore$ α is a 4th quadrant.

R.A. = $\tan^{-1}(1) = \pi/4$

$\therefore \alpha = 2\pi - \pi/4 = \underline{\dfrac{7\pi}{4}}$

$\therefore \cos x - \sin x = \underline{\sqrt{2}\cos\left(x - \frac{7\pi}{4}\right)}$

Max at: $\sqrt{2}\cos\left(x - \frac{7\pi}{4}\right) = \sqrt{2}$

$\cos\left(x - \frac{7\pi}{4}\right) = 1$

$x - \frac{7\pi}{4} = 0$

$x = \frac{7\pi}{4}$

$\therefore$ <u>Maximum Value of $\sqrt{2}$ at $x = \frac{7\pi}{4}$</u>

2000 – Higher II

1) at $x=1$: $y = 1 - 3 + 2 = 0$ $\quad\underline{\underline{(1,0)}}$

$M = \frac{dy}{dx} = 3x^2 - 6x + 2$

at $x=1$: $M = 3 - 6 + 2 = \underline{\underline{-1}}$

$y - b = M(x-a)$

$y - 0 = -1(x-1)$

$\underline{\underline{y = 1 - x}}$

b) $y = y$

$x^3 - 3x^2 + 2x = 2x - 4$

$x^3 - 3x^2 + 4 = 0$

$$\begin{array}{c|cccc} 2 & 1 & -3 & 0 & 4 \\ & & 2 & -2 & -4 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

$(x-2)(x^2 - x - 2) = 0$

$(x-2)(x-2)(x+1) = 0$

$\therefore \underline{\underline{x = -1}}$

$y = 2 \times -1 - 4 = \underline{\underline{-6}}$

$\therefore \underline{\underline{(-1,-6)}}$

2000 - Higher II

2) a)

Mid Point of PQ $(-1, 5)$

$M_{PQ} = \frac{8}{4} = \underline{\underline{2}}$

$\therefore M_{PQ} M_{AB} = -1 \qquad M_{AB} = \underline{\underline{\frac{-1}{2}}}$

$y - b = m(x - a)$

$y - 5 = -\frac{1}{2}(x + 1)$

$2y - 10 = -x - 1$

$\underline{\underline{2y + x - 9 = 0}}$

b) $x_c = 1 \qquad 2y = 9 - x$

$2y = 8$

$\underline{\underline{y = 4}} \qquad \therefore C(1, 4)$

$|CQ| = r = 9 - 4 = \underline{\underline{5}}$

$\therefore (x - a)^2 + (y - b)^2 = r^2$

$\underline{\underline{(x - 1)^2 + (y - 4)^2 = 25}}$

c) i) $\underline{\underline{y = 9}}$

ii) $2 \times 9 + x - 9 = 0$

$18 + x - 9 = 0$

$\underline{\underline{x = -9}} \qquad \underline{\underline{T(-9, 9)}}$

2000 – Higher II

3) a) $p(x) = 3 - \frac{3}{x}$

b) $p(q(x)) = 3 - \frac{3}{\frac{3}{3-x}}$

$= 3 - \frac{3(3-x)}{3} = 3 - 3 + x$

$= x$

4) a) $y = kx(x-4)$

at $(2,4)$: $4 = 2k(-2)$

$4 = -4k$

$k = -1 \quad \therefore \quad y = -x(x-4)$

$y = 4x - x^2$

b) $A = \int_2^k 4x - x^2 \, dx = \left[2x^2 - \frac{x^3}{3}\right]_2^k$

$= \left(2k^2 - \frac{k^3}{3}\right) - \left(8 - \frac{8}{3}\right)$

$= -\frac{1}{3}k^3 + 2k^2 - \frac{16}{3}$

5) $3\cos 2x + \cos x = -1$

$3(2\cos^2 x - 1) + \cos x = -1$

$6\cos^2 x - 3 + \cos x + 1 = 0$

$6\cos^2 x + \cos x - 2 = 0$

$a = 6, \ b = 1, \ c = -2$ ↗

$\therefore \cos x = 0.5$

$x = 60°$

$x = 300°$

$\cos x = -0.666$

RA $= 48.2$

$x = 180 - 48.2$

$= 131.8°$

$x = 180 + 48.2$

$= 228.2°$

$\therefore x = 60°, 131.8°, 228.2°, 300°$

2000- Higher II

6) $A(x) = \frac{3\sqrt{3}}{2}x^2 + 24\sqrt{3}\,x^{-1}$

Min at $A'(x)=0$: $3\sqrt{3}\,x - \frac{24\sqrt{3}}{x^2} = 0 \quad \times x^2$

$3\sqrt{3}\,x^3 - 24\sqrt{3} = 0$

$x^3 = \frac{24\sqrt{3}}{3\sqrt{3}}$

$x^3 = 8$

$\underline{\underline{x = 2}}$

x	2^-	2	2^+
$A'(x)$	$-Ve$	0	$+Ve$
SHAPE	\	—	/

$\therefore$ $\underline{\underline{\text{Min at } x=2}}$

7) If perpendicular $a \cdot b = 0$

$u \cdot v = 2t - 20 + 3t = 0$

$5t = 20$

$\underline{\underline{t = 4}}$

8) $f'(x) = \frac{1}{2}(5x-4)^{-1/2} \times 5$

$= \frac{5}{2\sqrt{5x-4}}$

$f'(4) = \frac{5}{2 \times \sqrt{16}} = \underline{\underline{\frac{5}{8}}}$

2000 - Higher II

9) a) $(3, 2, 15)$

b) $\overrightarrow{BA} = \begin{pmatrix} 0 \\ 9 \\ 8 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 15 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \\ -7 \end{pmatrix}$ $\qquad \overrightarrow{BC} = \begin{pmatrix} 17 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 15 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \\ -7 \end{pmatrix}$

$$\cos\theta = \frac{a.b}{|a|\,|b|}$$

$$|\overrightarrow{BA}| = \sqrt{9+49+49} = \underline{\sqrt{107}}$$

$$|\overrightarrow{BC}| = \sqrt{196+4+49} = \underline{\underline{\sqrt{249}}}$$

$$\overrightarrow{BA}.\overrightarrow{BC} = (-3\times14)+(7\times-2)+(-7\times-7) = \underline{\underline{-7}}$$

$$\therefore \cos\theta = \frac{-7}{\sqrt{107}\times\sqrt{249}} = \underline{\underline{-0.043}}$$

$$\therefore \angle ABC = \cos^{-1}(-0.043) = \underline{\underline{92.5^\circ}}$$

10) $$\int (7-3x)^{-2}\,dx = \frac{1}{-3(-2+1)}(7-3x)^{-2+1} + C$$

$$= \frac{1}{3}(7-3x)^{-1} + C$$

$$= \underline{\underline{\frac{1}{3(7-3x)}}} + C$$

2000 - Higher II

11) a) $\underline{P = 0.6Q + 1.8}$ $M = \frac{1.8}{3} = 0.6$

b) $P = aq^b$

$\log_e P = \log_e aq^b$

$\log_e P = \log_e a + \log_e q^b$

$\log_e P = \log_e a + b \log_e q$

$\log_e P = b\log_e q + \log_e a$

$P = bQ + \log_e a$

and $P = 0.6Q + 1.8$

$\therefore \underline{b = 0.6}$

$\log_e a = 1.8$

$a = e^{1.8}$

$a = \underline{6.05}$