SCOTTISH CERTIFICATE OF EDUCATION 1999

MONDAY, 10 MAY
9.15 AM - 11.15 AM

MATHEMATICS
HIGHER GRADE
Paper I

All questions should be attempted *Marks*

1. (a) Show that $x = 2$ is a root of the equation $2x^3 + x^2 - 13x + 6 = 0$. **(1)**

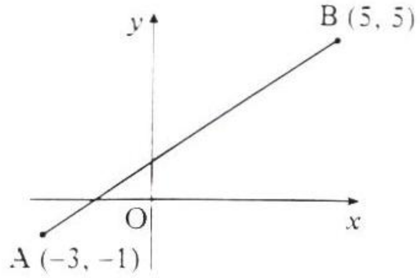
   (b) Hence find the other roots. **(3)**

2. A and B are the points $(-3, -1)$ and $(5, 5)$.

   Find the equation of

   (a) the line AB **(2)**

   (b) the perpendicular bisector of AB. **(3)**

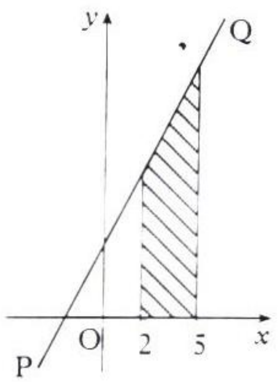


3. The line PQ has equation $y = 2x + 4$.

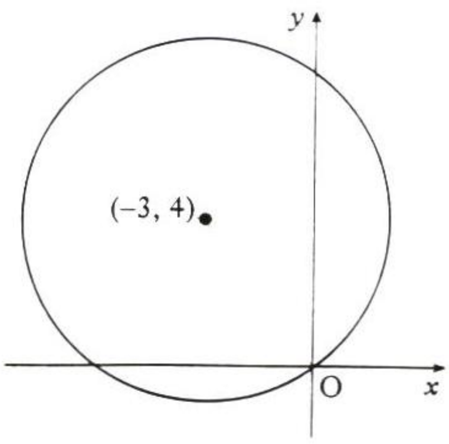
   (a) Find, without using calculus, the area of the shaded trapezium shown in the diagram. **(2)**

   (b) Express the area of this trapezium as a definite integral. **(1)**

   (c) Evaluate this integral. **(2)**



4. Find the equation of the circle with centre $(-3, 4)$ and passing through the origin. **(2)**



Marks

5. Given $f(x) = 3x^2(2x - 1)$, find $f'(-1)$. **(3)**

6. VABCD is a pyramid with rectangular base ABCD.

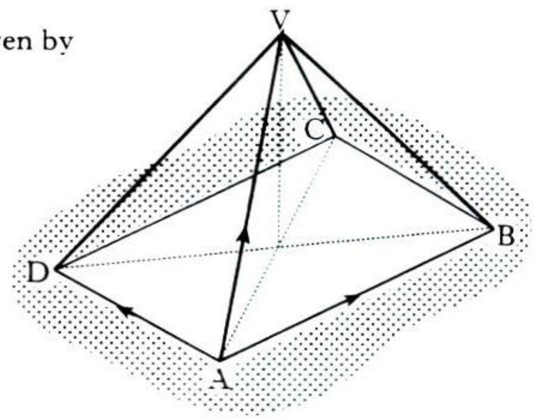
The vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AV}$ are given by

$$\overrightarrow{AB} = 8\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$$

$$\overrightarrow{AD} = -2\boldsymbol{i} + 10\boldsymbol{j} - 2\boldsymbol{k} \quad \text{and}$$

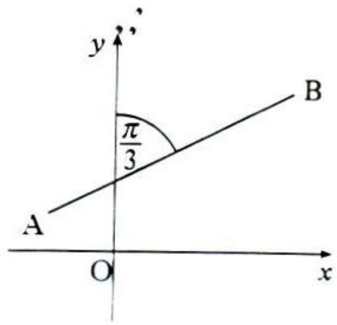
$$\overrightarrow{AV} = \boldsymbol{i} + 7\boldsymbol{j} + 7\boldsymbol{k}.$$

Express $\overrightarrow{CV}$ in component form.



(3)

7. The line AB makes an angle of $\frac{\pi}{3}$ radians with the y-axis, as shown in the diagram. Find the exact value of the gradient of AB.



(2)

8. (i) Write down the condition for the equation $ax^2 + bx + c = 0$ to have equal roots.

(ii) Hence, or otherwise, show that the equation $x(x + 7) = x - 9$ has equal roots. **(3)**

9. The point P(–1, 7) lies on the curve with equation $y = 5x^2 + 2$. Find the equation of the tangent to the curve at P. **(4)**

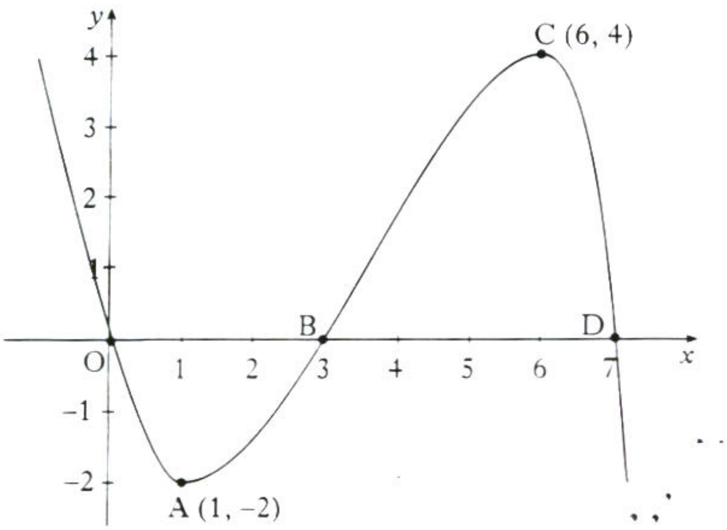
Marks

10. Part of the graph of $y = f(x)$ is shown in the diagram. On separate diagrams, sketch the graph of

(*a*) $y = f(x + 1)$ **(2)**

(*b*) $y = -2f(x)$. **(3)**

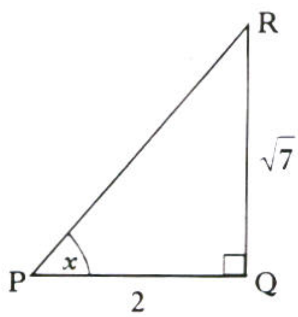
Indicate on each graph the images of O, A, B, C and D.



11. The graph of $y = g(x)$ passes through the point $(1, 2)$.

If $\frac{dy}{dx} = x^3 + \frac{1}{x^2} - \frac{1}{4}$, express y in terms of x. **(4)**

12. Using triangle **PQR**, as shown, find the exact value of $\cos 2x$. **(3)**



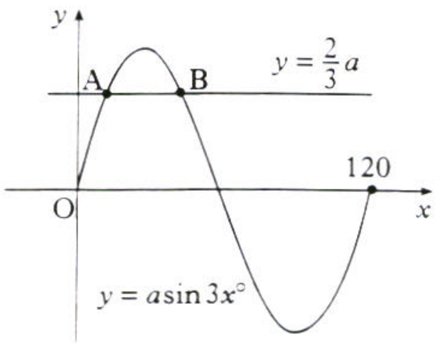
Marks

13. (*a*) Show that $f(x) = 2x^2 - 4x + 5$ can be written in the form $f(x) = a(x + b)^2 + c$. **(3)**

(*b*) Hence write down the coordinates of the stationary point of $y = f(x)$ and state its nature. **(2)**

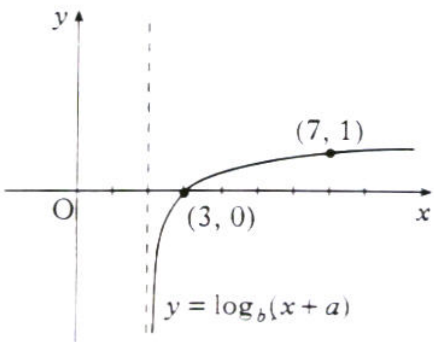
14. The diagram shows part of the graph of $y = a\sin 3x°$ and the line with equation $y = \frac{2}{3}a$.

Find the x-coordinates of A and B. **(4)**



15. The diagram shows part of the graph of $y = \log_b(x + a)$.

Determine the values of a and b. **(3)**

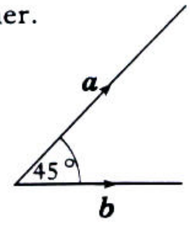


16. A curve has equation $y = 2x^3 + 3x^2 + 4x - 5$.

Prove that this curve has no stationary points. **(5)**

17. The diagram shows two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, with $|\boldsymbol{a}| = 3$ and $|\boldsymbol{b}| = 2\sqrt{2}$.

These vectors are inclined at an angle of 45° to each other.



(*a*) Evaluate (i) $\boldsymbol{a}.\boldsymbol{a}$

(ii) $\boldsymbol{b}.\boldsymbol{b}$

(iii) $\boldsymbol{a}.\boldsymbol{b}$ **(2)**

(*b*) Another vector $\boldsymbol{p}$ is defined by $\boldsymbol{p} = 2\boldsymbol{a} + 3\boldsymbol{b}$.

Evaluate $\boldsymbol{p}.\boldsymbol{p}$ and hence write down $|\boldsymbol{p}|$. **(4)**

18. Two sequences are defined by the recurrence relations

$$u_{n+1} = 0 \cdot 2u_n + p, \quad u_0 = 1 \quad \text{and}$$
$$v_{n+1} = 0 \cdot 6v_n + q, \quad v_0 = 1.$$

If both sequences have the same limit, express p in terms of q. **(3)**

19. Given $f(x) = \cos^2 x - \sin^2 x$, find $f'(x)$. **(3)**

20. Find $\int \frac{x^2 - 5}{x\sqrt{x}} dx$. **(4)**

21. A function f can be expressed as an infinite series by

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$$

(*a*) Write down the series for $f(2x)$ as far as the term in x^5. **(1)**

The derivative of $f(x)$ can be calculated as follows.

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$$

$$\text{so} \quad f'(x) = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \frac{5x^4}{120} + \ldots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ldots$$

$$\text{ie} \quad f'(x) = f(x)$$

(*b*) Find $f'(2x)$ in terms of $f(2x)$. **(3)**

[*END OF QUESTION PAPER*]

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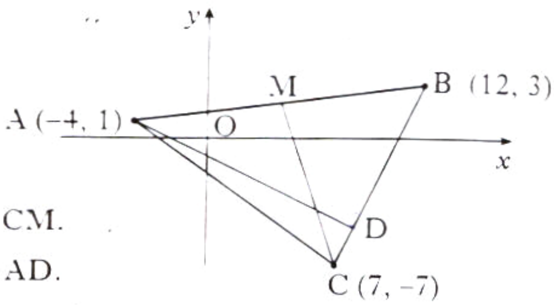
MONDAY, 10 MAY
1.00 PM – 3.30 PM

MATHEMATICS
HIGHER GRADE
Paper II

All questions should be attempted *Marks*

1. A triangle ABC has vertices A $(-4, 1)$, B $(12, 3)$ and C $(7, -7)$.



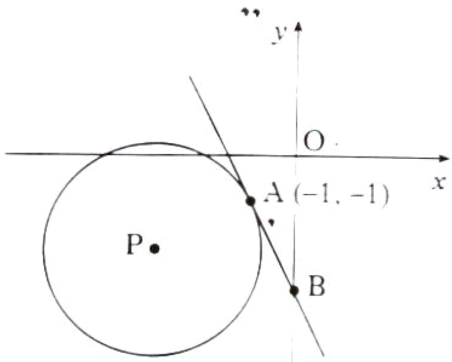
(*a*) Find the equation of the median CM. **(3)**

(*b*) Find the equation of the altitude AD. **(3)**

(*c*) Find the coordinates of the point of intersection of CM and AD. **(3)**

2. (*a*) The diagram shows a circle, centre P, with equation $x^2 + y^2 + 6x + 4y + 8 = 0$.

Find the equation of the tangent at the point A $(-1, -1)$ on the circle. **(4)**

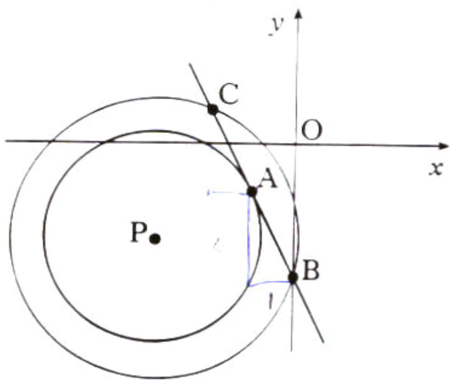


(*b*) The tangent crosses the y-axis at B.

Find the coordinates of B. **(1)**

(*c*) Another circle, centre P, is drawn passing through B. The tangent at A meets the second circle at the point C, as shown in the diagram.

Write down the coordinates of C. **(1)**



(*d*) Find the equation of the circle with BC as diameter. **(2)**

Marks

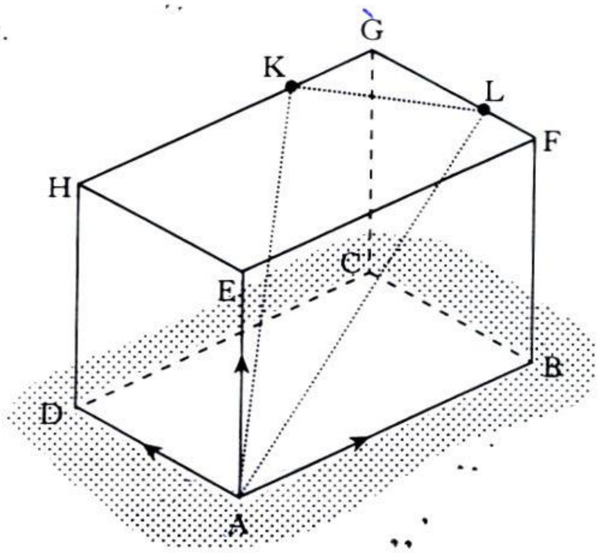
3. ABCDEFGH is a cuboid.

K lies two thirds of the way along HG, (ie HK:KG = 2:1).

L lies one quarter of the way along FG, (ie FL:LG = 1:3).

$\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AE}$ can be represented by the vectors

$\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ respectively.

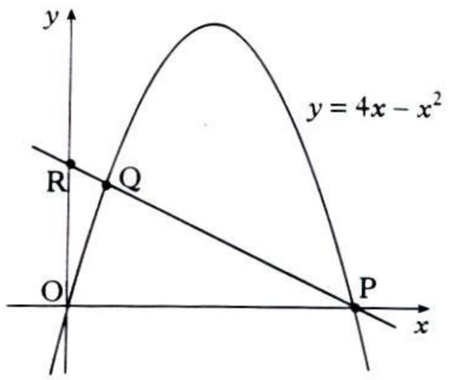


(*a*) Calculate the components of $\overrightarrow{AK}$. **(2)**

(*b*) Calculate the components of $\overrightarrow{AL}$. **(2)**

(*c*) Calculate the size of angle KAL. **(5)**

4. The parabola shown in the diagram has equation $y = 4x - x^2$ and intersects the x-axis at the origin and P.



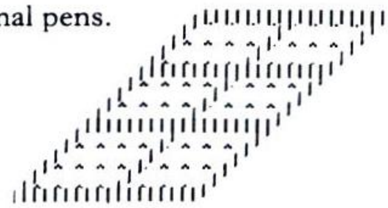
(*a*) Find the coordinates of the point P. **(2)**

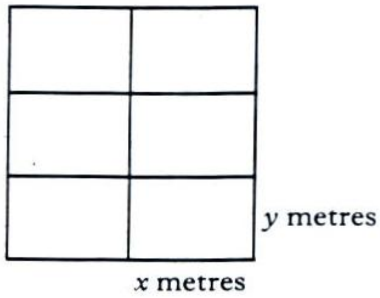
(*b*) R is the point (0, 2). Find the equation of PR. **(2)**

(*c*) The line and the parabola also intersect at Q. Find the coordinates of Q. **(4)**

Marks

5. A zookeeper wants to fence off six individual animal pens.





Each pen is a rectangle measuring x metres by y metres, as shown in the diagram.

(*a*) (i) Express the total length of fencing in terms of x and y.

(ii) Given that the total length of fencing is 360 m, show that the total area, $A\,\text{m}^2$, of the six pens is given by $A(x) = 240x - \frac{16}{3}x^2$. **(4)**

(*b*) Find the values of x and y which give the maximum area and write down this maximum area. **(6)**

6. Functions f and g are defined on the set of real numbers by

$$f(x) = x - 1$$
$$g(x) = x^2.$$

(*a*) Find formulae for

(i) $f(g(x))$

(ii) $g(f(x))$. **(4)**

(*b*) The function h is defined by $h(x) = f(g(x)) + g(f(x))$.
Show that $h(x) = 2x^2 - 2x$ and sketch the graph of h. **(3)**

(*c*) Find the area enclosed between this graph and the x-axis. **(4)**

[Turn over

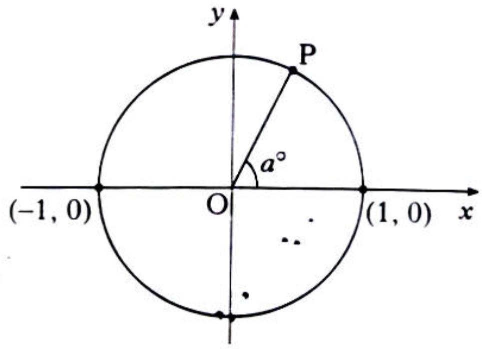
Marks

7. The intensity I_t of light is reduced as it passes through a filter according to the law $I_t = I_0 e^{-kt}$ where I_0 is the initial intensity and I_t is the intensity after passing through a filter of thickness t cm. k is a constant.

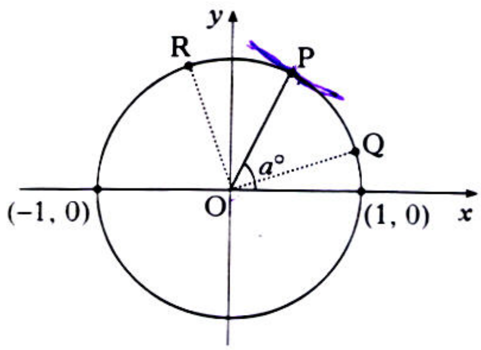
(*a*) A filter of thickness 4 cm reduces the intensity from 120 candle-power to 90 candle-power. Find the value of k. **(4)**

(*b*) Light is passed through a filter of thickness 10 cm. Find the percentage reduction in its intensity. **(3)**

8. The diagram shows a circle of radius 1 unit and centre the origin. The radius OP makes an angle $a°$ with the positive direction of the x-axis.



(*a*) Show that P is the point $(\cos a°, \sin a°)$. **(1)**



(*b*) If $P\hat{O}Q = 45°$, deduce the coordinates of Q in terms of a. **(1)**

(*c*) If $P\hat{O}R = 45°$, deduce the coordinates of R in terms of a. **(1)**

(*d*) Hence find an expression for the gradient of QR in its simplest form. **(4)**

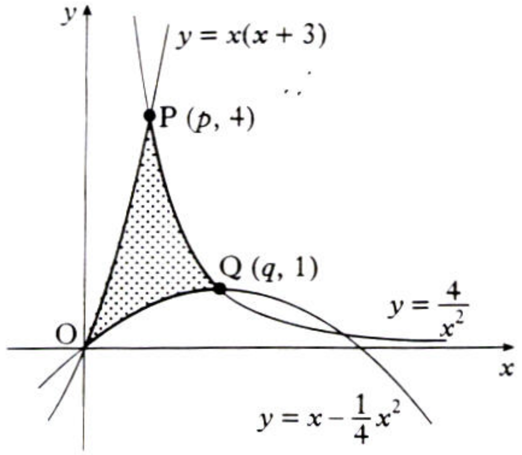
(*e*) Show that the tangent to the circle at P is parallel to QR. **(2)**

9. Solve the equation $2\sin x° - 3\cos x° = 2\cdot5$ in the interval $0 \leq x < 360$. **(8)**

Marks

10. The origin, O, and the points P and Q are the vertices of a curved "triangle" which is shaded in the diagram.

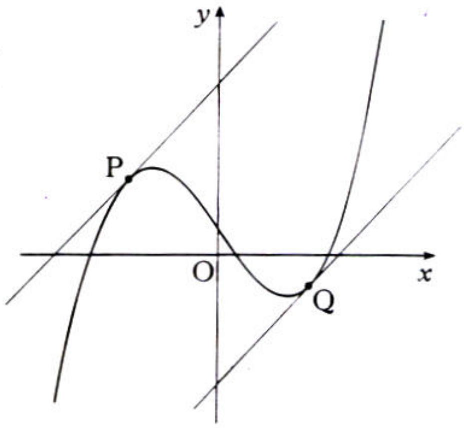
The sides lie on curves with equations $y = x(x+3)$, $y = x - \frac{1}{4}x^2$ and $y = \frac{4}{x^2}$.



(*a*) P and Q have coordinates $(p, 4)$ and $(q, 1)$. Find the values of p and q. **(2)**

(*b*) Calculate the shaded area. **(7)**

11. The diagram shows a sketch of the graph of $y = x^3 - 9x + 4$ and two parallel tangents drawn at P and Q.



(*a*) Find the equations of the tangents to the curve $y = x^3 - 9x + 4$ which have gradient 3. **(6)**

(*b*) Show that the shortest distance between the tangents is $\frac{16\sqrt{10}}{5}$. **(6)**

[END OF QUESTION PAPER]