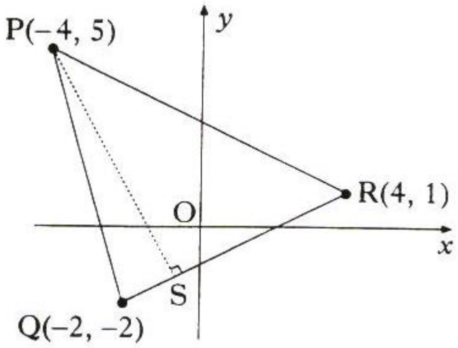
SCOTTISH CERTIFICATE OF EDUCATION 1997

THURSDAY, 8 MAY
9.30 AM – 11.30 AM

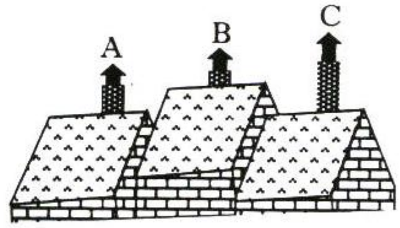
MATHEMATICS
HIGHER GRADE
Paper I

All questions should be attempted *Marks*

1. P(−4, 5), Q(−2, −2) and R(4, 1) are the vertices of triangle PQR as shown in the diagram. Find the equation of PS, the altitude from P. **(3)**



2. Relative to a suitable set of axes, the tops of three chimneys have coordinates given by A(1, 3, 2), B(2, −1, 4) and C(4, −9, 8).
Show that A, B and C are collinear. **(3)**



3. Functions f and g, defined on suitable domains, are given by $f(x) = 2x$ and $g(x) = \sin x + \cos x$.
Find $f(g(x))$ and $g(f(x))$. **(4)**

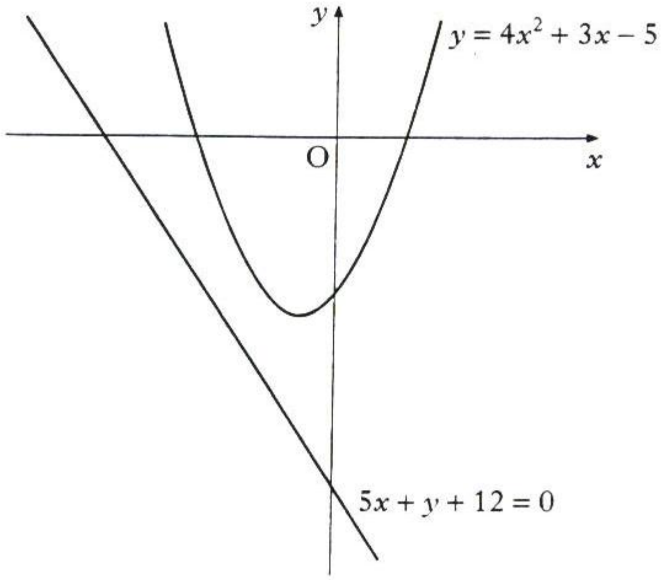
4. The position vectors of the points P and Q are $\boldsymbol{p} = -\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{q} = 7\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ respectively.
   (*a*) Express $\overrightarrow{PQ}$ in component form. **(2)**
   (*b*) Find the length of PQ. **(1)**

5. (*a*) Find a real root of the equation $2x^3 - 3x^2 + 2x - 8 = 0$. **(2)**
   (*b*) Show algebraically that there are no other real roots. **(3)**

6. The diagram below shows a parabola with equation $y = 4x^2 + 3x - 5$ and a straight line with equation $5x + y + 12 = 0$.

A tangent to the parabola is drawn parallel to the given straight line.

Find the x-coordinate of the point of contact of this tangent. **(5)**



7. If $x°$ is an acute angle such that $\tan x° = \frac{4}{3}$, show that the exact value of $\sin(x + 30)°$ is $\frac{4\sqrt{3} + 3}{10}$. **(3)**

8. Given that $y = 2x^2 + x$, find $\frac{dy}{dx}$ and hence show that $x\left(1 + \frac{dy}{dx}\right) = 2y$. **(3)**

9. (*a*) Show that the function $f(x) = 2x^2 + 8x - 3$ can be written in the form $f(x) = a(x + b)^2 + c$ where a, b and c are constants. **(3)**

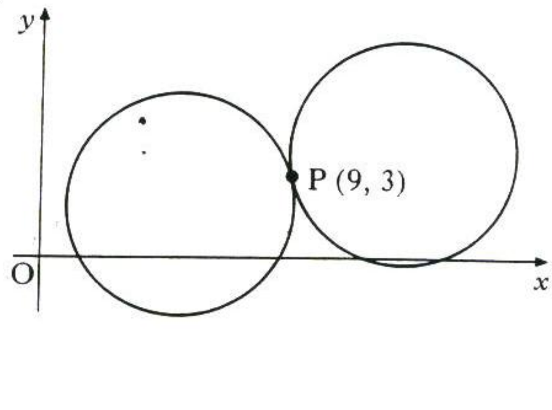
   (*b*) Hence, or otherwise, find the coordinates of the turning point of the function f. **(1)**

10. Find the value of $\int_1^4 \sqrt{x}\,dx$. **(4)**

11. Express $2\sin x° - 5\cos x°$ in the form $k\sin(x - \alpha)°$, $0 \leq \alpha < 360$ and $k > 0$. **(4)**

12. Two identical circles touch at the point P (9, 3) as shown in the diagram. One of the circles has equation $x^2 + y^2 - 10x - 4y + 12 = 0$.

Find the equation of the other circle. **(5)**

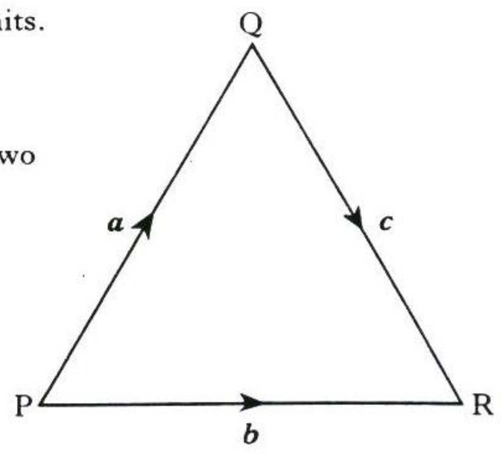


Marks

NC **13.** PQR is an equilateral triangle of side 2 units.

$\overrightarrow{PQ} = \boldsymbol{a}$, $\overrightarrow{PR} = \boldsymbol{b}$ and $\overrightarrow{QR} = \boldsymbol{c}$.

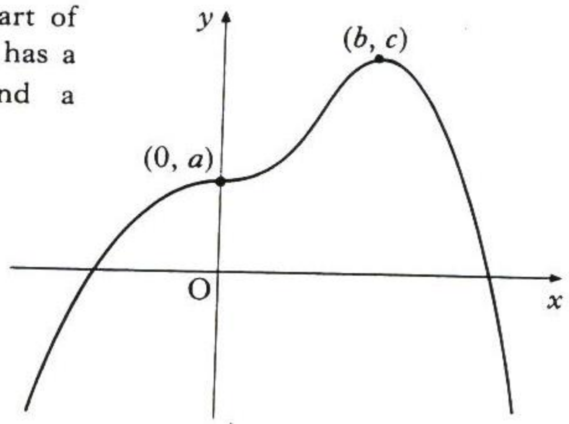
Evaluate $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c})$ and hence identify two vectors which are perpendicular. **(4)**



14. For what range of values of c does the equation $x^2 + y^2 - 6x + 4y + c = 0$ represent a circle? **(3)**

15. The curve $y = f(x)$ passes through the point $\left(\frac{\pi}{12}, 1\right)$ and $f'(x) = \cos 2x$. Find $f(x)$. **(3)**

16. The diagram shows a sketch of part of the graph of $y = f(x)$. The graph has a point of inflection at $(0, a)$ and a maximum turning point at (b, c).



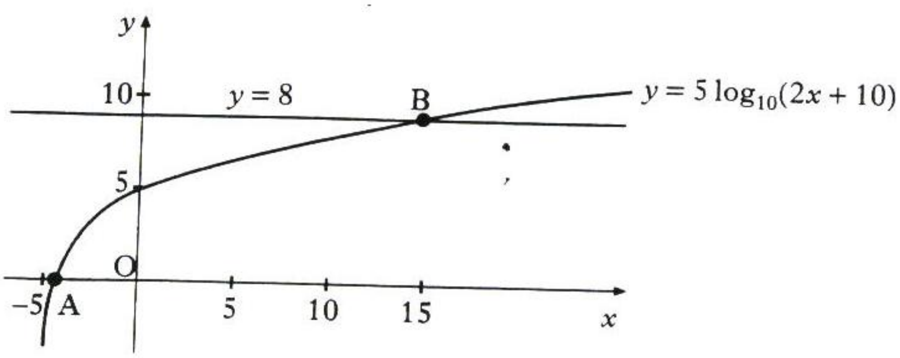
(a) Make a copy of this diagram and on it sketch the graph of $y = g(x)$ where $g(x) = f(x) + 1$. **(2)**

(b) On a separate diagram, sketch the graph of $y = f'(x)$. **(2)**

(c) Describe how the graph of $y = g'(x)$ is related to the graph of $y = f'(x)$. **(1)**

17. Part of the graph of $y = 5\log_{10}(2x + 10)$ is shown in the diagram. This graph crosses the x-axis at the point A and the straight line $y = 8$ at the point B.

Find algebraically the x-coordinates of A and B. **(4)**

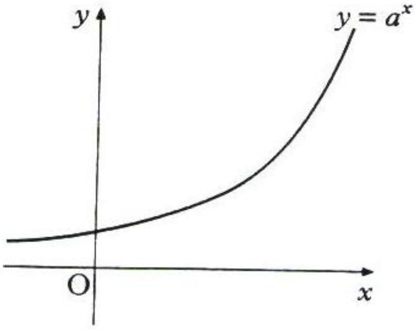


18. (*a*) Show that $2\cos 2x° - \cos^2 x° = 1 - 3\sin^2 x°$. **(2)**

(*b*) **Hence** solve the equation

$$2\cos 2x° - \cos^2 x° = 2\sin x° \text{ in the interval } 0 \le x < 360.$$ **(4)**

19. The diagram shows a sketch of part of the graph of $y = a^x$, $a > 1$.



(*a*) If $(1, t)$ and $(u, 1)$ lie on this curve, write down the values of t and u. **(2)**

(*b*) Make a copy of this diagram and on it sketch the graph of $y = a^{2x}$. **(2)**

(*c*) Find the coordinates of the point of intersection of $y = a^{2x}$ with the line $x = 1$. **(1)**

20. Diagram 1 shows 5 cars travelling up an incline on a roller-coaster. Part of the roller-coaster rail follows the curve with equation $y = 8 + 5\cos\frac{1}{2}x$.

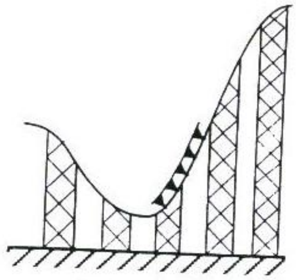


Diagram 1

Diagram 2 shows an enlargement of the last car and its position relative to a suitable set of axes. The floor of the car lies parallel to the tangent at P, the point of contact.

Calculate the acute angle a between the floor of the car and the horizontal when the car is at the point where $x_p = \frac{7\pi}{3}$.

Express your answer in degrees. **(4)**

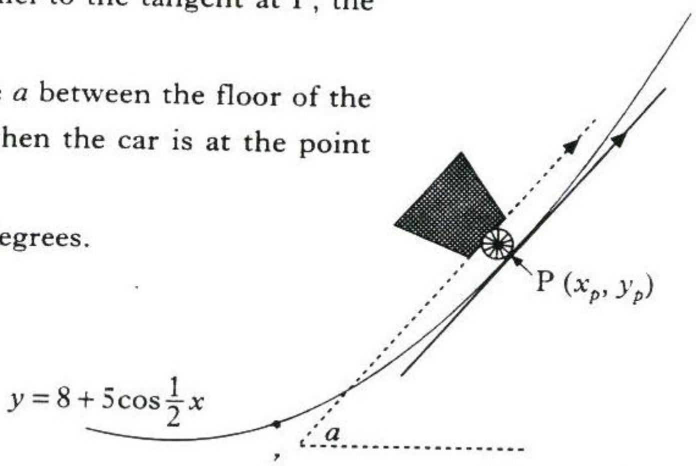


Diagram 2

[*END OF QUESTION PAPER*]

SCOTTISH CERTIFICATE OF EDUCATION 1997

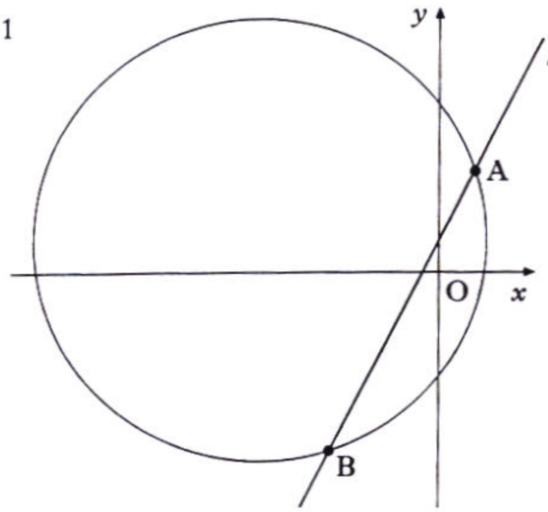
THURSDAY, 8 MAY
1.00 PM – 3.30 PM

MATHEMATICS
HIGHER GRADE
Paper II

All questions should be attempted *Marks*

1. Diagram 1 shows a circle with equation $x^2 + y^2 + 10x - 2y - 14 = 0$ and a straight line, l_1, with equation $y = 2x + 1$.
The line intersects the circle at A and B.

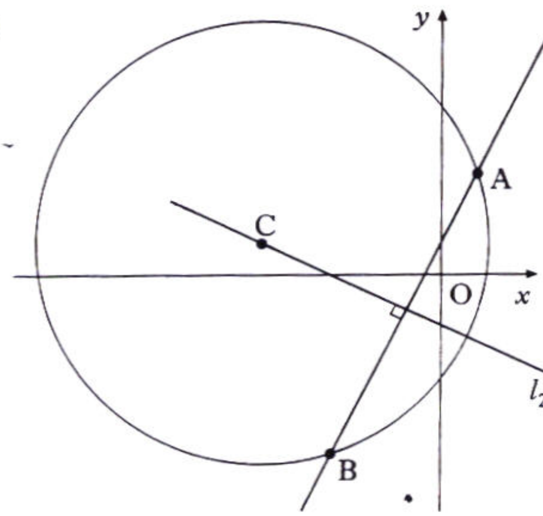
Diagram 1



(a) Find the coordinates of the points A and B. **(5)**

(b) Diagram 2 shows a second line, l_2, which passes through the centre of the circle, C, and is at right angles to line l_1.

Diagram 2



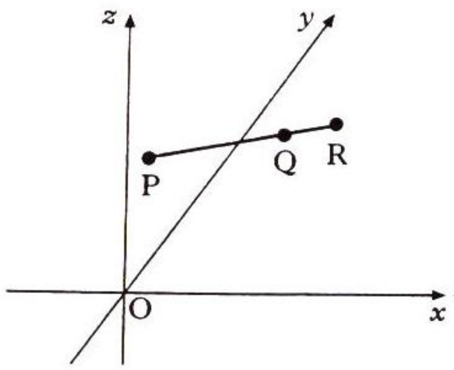
(i) Write down the coordinates of C. **(1)**

(ii) Find the equation of the line l_2. **(3)**

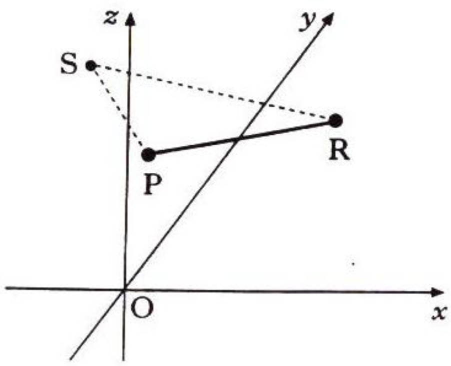
Marks

2. Relative to the axes shown and with an appropriate scale, P(−1, 3, 2) and Q(5, 0, 5) represent points on a road. The road is then extended to the point R such that $\overrightarrow{PR} = \frac{4}{3}\overrightarrow{PQ}$.

(a) Find the coordinates of R. **(3)**



(b) Roads from P and R are built to meet at the point S (−2, 2, 5). Calculate the size of angle PSR. **(7)**



3. The sum of £1000 is placed in an investment account on January 1st and, thereafter, £100 is placed in the account on the first day of each month.

- Interest at the rate of 0·5% per month is credited to the account on the last day of each month.
- This interest is calculated on the amount in the account on the first day of the month.

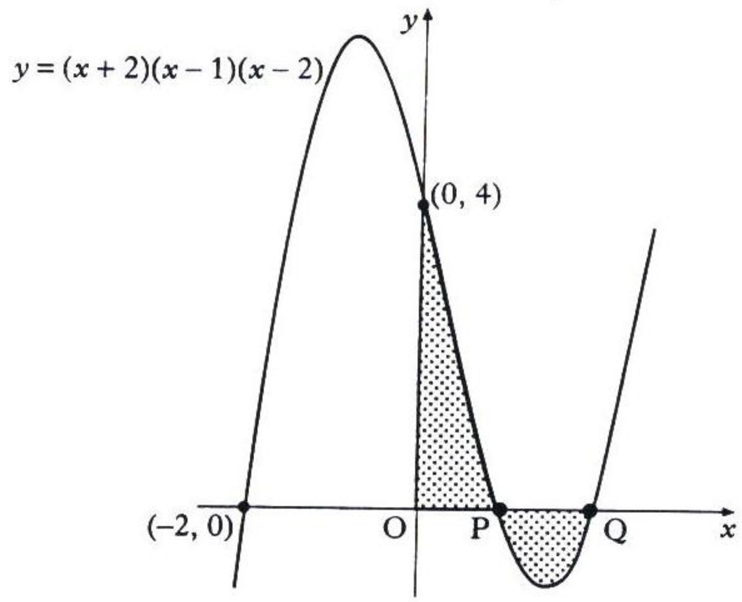
(a) How much is in the account on June 30th? **(4)**

(b) On what date does the account first exceed £2000? **(2)**

(c) Find a recurrence relation which describes the amount in the account, explaining your notation carefully. **(3)**

Marks

4. The diagram shows a sketch of the graph of $y = (x + 2)(x - 1)(x - 2)$. The graph cuts the axes at $(-2, 0)$, $(0, 4)$ and the points P and Q.



(a) Write down the coordinates of P and Q. **(2)**

(b) Find the total shaded area. **(7)**

[Turn over

Marks

5. Diagram 1 shows a sketch of part of the graph of $y = f(x)$ where $f(x) = (x-2)^2 + 1$.

The graph cuts the y-axis at A and has a minimum turning point at B.

(*a*) Write down the coordinates of A and B. **(3)**

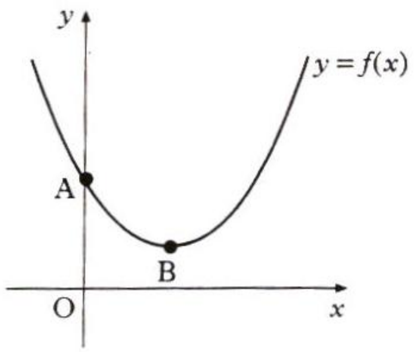


Diagram 1

(*b*) Diagram 2 shows the graphs of $y = f(x)$ and $y = g(x)$ where $g(x) = 5 + 4x - x^2$.

Find the area enclosed by the two curves. **(5)**

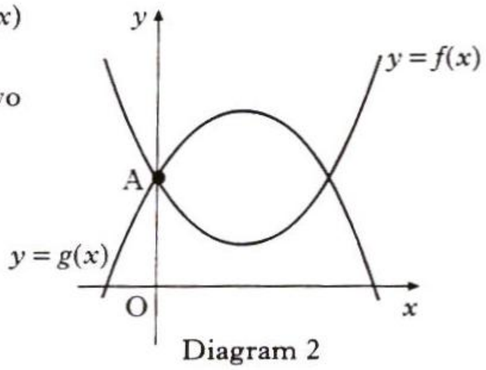


Diagram 2

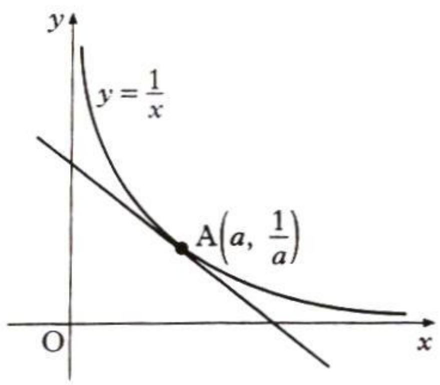
(*c*) $g(x)$ can be written in the form $m + n \times f(x)$ where m and n are constants.

Write down the values of m and n. **(2)**

6. (*a*) A sketch of part of the graph of $y = \frac{1}{x}$ is shown in the diagram.

The tangent at $A\left(a, \frac{1}{a}\right)$ has been drawn.

Find the gradient of this tangent. **(4)**



(*b*) Hence show that the equation of this tangent is $x + a^2y = 2a$. **(2)**

(*c*) This tangent cuts the y-axis at B and the x-axis at C.

(i) Calculate the area of triangle OBC. **(3)**

(ii) Comment on your answer to c(i). **(1)**

7. In certain topics in Mathematics, such as calculus, we often require to write an expression such as $\frac{8x+1}{(2x+1)(x-1)}$ in the form $\frac{2}{2x+1}+\frac{3}{x-1}$.

$\frac{2}{2x+1}+\frac{3}{x-1}$ are called **Partial Fractions** for $\frac{8x+1}{(2x+1)(x-1)}$.

The worked example shows you how to find partial fractions for the expression $\frac{6x+2}{(x+2)(x-3)}$.

Worked Example

Find partial fractions for $\frac{6x+2}{(x+2)(x-3)}$.

Let $\frac{6x+2}{(x+2)(x-3)} = \frac{A}{x+2}+\frac{B}{x-3}$ where A and B are constants

$$= \frac{A(x-3)}{(x+2)(x-3)}+\frac{B(x+2)}{(x-3)(x+2)}$$

ie $\frac{6x+2}{(x+2)(x-3)} = \frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$

Hence $6x+2 = A(x-3)+B(x+2)$ for all values of x.

A and B can be found as follows:

Select a value of x that makes the first bracket zero

Let $x = 3$ (this eliminates A)

$18+2 = A\times 0 + B\times 5$

$20 = 5B$

$\underline{B = 4}$

Select a value of x that makes the second bracket zero

Let $x = -2$ (this eliminates B)

$-12+2 = A\times(-5) + B\times 0$

$-10 = -5A$

$\underline{A = 2}$

Therefore $\frac{6x+2}{(x+2)(x-3)} = \frac{2}{x+2}+\frac{4}{x-3}$.

Find partial fractions for $\frac{5x+1}{(x-4)(x+3)}$. **(6)**

Marks

8. The radioactive element carbon-14 is sometimes used to estimate the age of organic remains such as bones, charcoal and seeds.

Carbon-14 decays according to a law of the form $y = y_0e^{kt}$ where y is the amount of radioactive nuclei present at time t years and y_0 is the initial amount of radioactive nuclei.

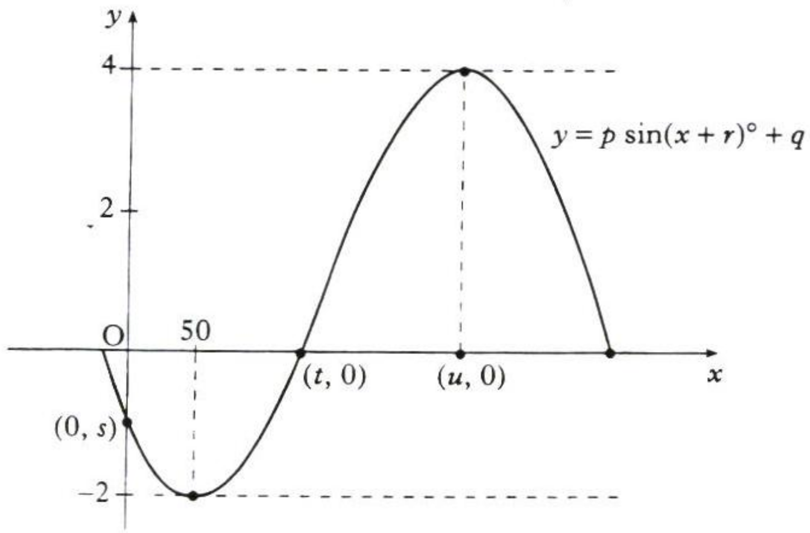
(*a*) The half-life of carbon-14, ie the time taken for half the radioactive nuclei to decay, is 5700 years. Find the value of the constant k, correct to 3 significant figures. **(3)**

(*b*) What percentage of the carbon-14 in a sample of charcoal will remain after 1000 years? **(3)**

9. The sketch represents part of the graph of a trigonometric function of the form $y = p\sin(x + r)° + q$. It crosses the axes at $(0, s)$ and $(t, 0)$, and has turning points at $(50, -2)$ and $(u, 4)$.

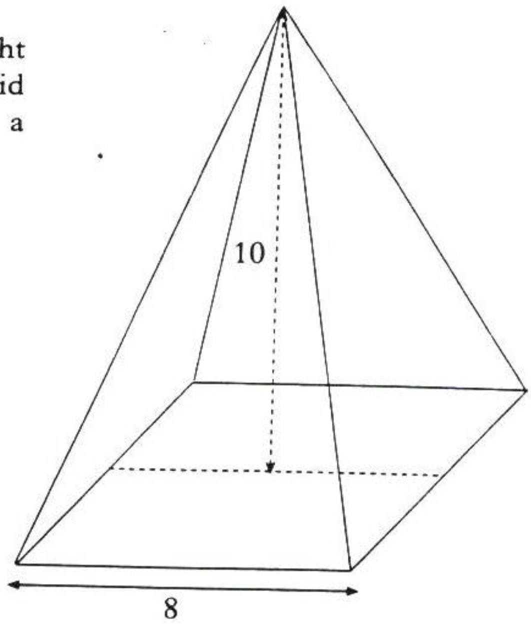
(*a*) Write down values for p, q, r and u. **(4)**

(*b*) Find the values for s and t. **(4)**

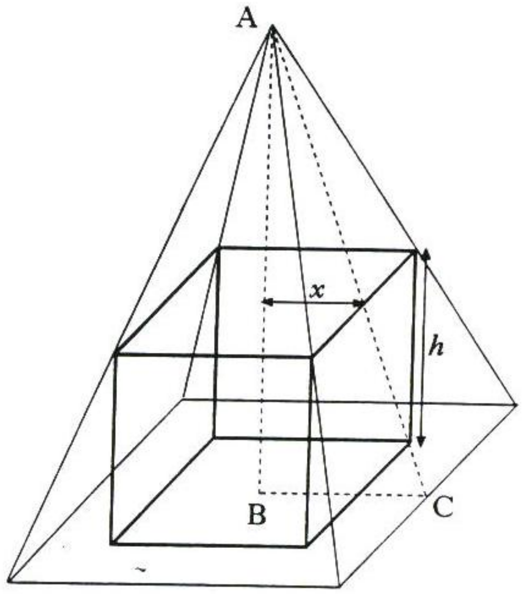


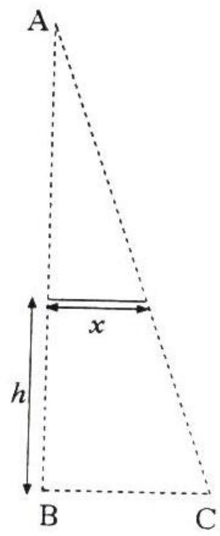
Marks

10. A cuboid is to be cut out of a right square-based pyramid. The pyramid has a square base of side 8 cm and a vertical height of 10 cm.



(*a*) The cuboid has a square base of side $2x$ cm and a height of h cm.





If the cuboid is to fit into the pyramid, use the information shown in triangle ABC, or otherwise, to show that:

(i) $h = 10 - \frac{5}{2}x$; **(3)**

(ii) the volume, V, of the cuboid is given by $V = 40x^2 - 10x^3$. **(1)**

(*b*) Hence, find the dimensions of the square-based cuboid with the greatest volume which can be cut from the pyramid. **(6)**

Marks

11. Two identical coins, radius 1 unit, are supported by horizontal and vertical plates at B and C. Diagram 1 shows the coins touching each other and the line of centres is inclined at p radians to the vertical.

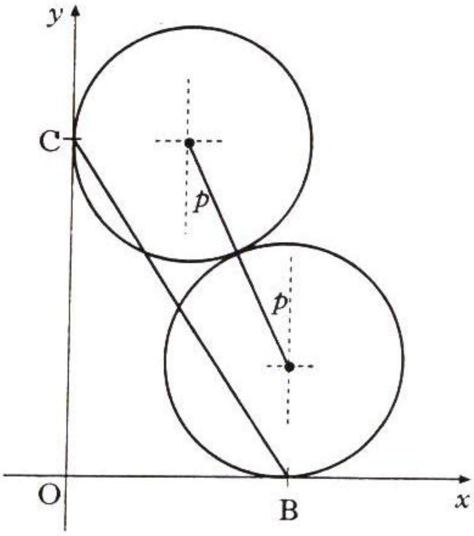


Diagram 1

Let d be the length of BC.

(*a*) (i) Show that $OB = 1 + 2\sin p$. **(1)**

(ii) Write down a similar expression for OC and hence show that $d^2 = 6 + 4\cos p + 4\sin p$. **(2)**

(*b*) (i) Express d^2 in the form $6 + k\cos(p - \alpha)$. **(4)**

(ii) Hence, write down the exact maximum value of d^2 and the value of p for which this occurs. **(2)**

(*c*) Diagram 2 shows the special case where $p = \frac{\pi}{4}$.

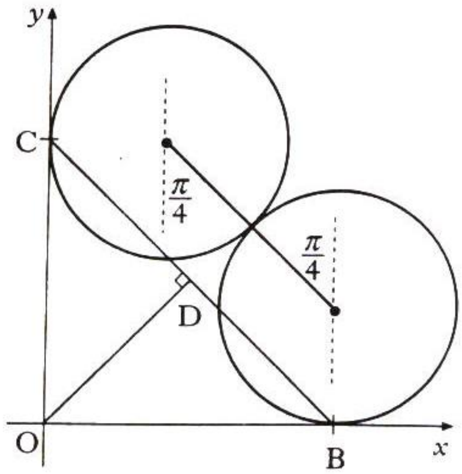


Diagram 2

(i) Show that $OB = 1 + \sqrt{2}$ and find the exact length of BD. **(2)**

(ii) Using your answer to (*b*) (ii), find the exact value of $\sqrt{6 + 4\sqrt{2}}$. **(2)**

[*END OF QUESTION PAPER*]